

# MODELLING SEASON CHANGES IN THE INFINITE PROCESSES OF GROWTH OF BIOLOGICAL SYSTEMS

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## Abstract

The paper introduces a new type of objects – blinking fractals – that are not covered by traditional theories considering dynamics of self-similarity processes. It is shown that both traditional and blinking fractals can be studied by a recent approach allowing one to work with infinite and infinitesimal numbers. It is shown that blinking fractals can be successfully applied for modelling complex processes of growth of biological systems including their season changes. The new approach allows one to give various quantitative characteristics of the obtained blinking fractals models of biological systems.

**Key Words:** Process of growth, biological models, blinking fractals, infinite and infinitesimal numbers, numerical computations.

## 1 Introduction

Fractals have been very well studied during the last few decades and have been used in various scientific fields including biology to model complex systems (see numerous applications given in [3, 4, 6, 9, 17]). The fractal objects are constructed by using the principle of self-similarity: a given basic figure (some times slightly

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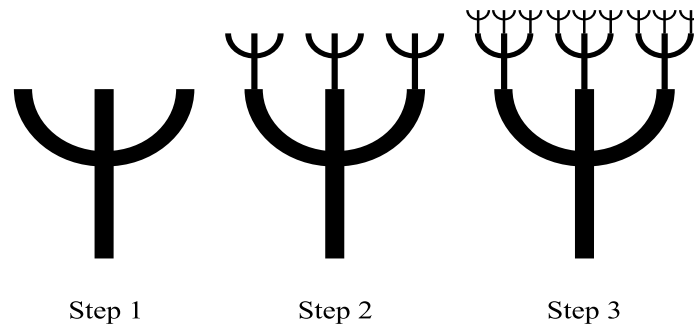


Figure 1: A simple fractal model of a tree.

modified in time) infinitely many times repeats itself in several copies. An example of such a construction is shown in Fig. 1. The basic figure shown in Step 1 is then repeated and already at Step 3 can be viewed as a simple model of a tree.

The introduction of fractals has allowed people to describe complex systems having a fractal structure in an elegant and very efficient way, to construct their computational models, and to study them. However, it is important to mention that mathematical analysis of fractals (except, of course, a very well developed theory of fractal dimensions, see [3, 4, 6, 9]) very often continues to have mainly a qualitative character. For example, tools for a quantitative analysis of fractals at infinity are not very rich yet (e.g., even for one of the mostly studied fractals – Cantor’s set – we are not able to count the number of intervals composing the set at infinity).

Nowadays, fractals usually are used to describe objects (see, e.g., a tree presented in Fig. 1, Step 3) where *one* basic figure often called *generator* can be determined; they are rarely used for modelling processes where appearance of the studied objects is changed in time without preserving the generator. For example, the model from Fig. 1 cannot be used to describe a tree if we take into account season changes because in summer the tree has green leafs, in autumn the leafs are yellow, in winter there are no leafs at all and branches of the tree are under the snow. Thus, we are not able to distinguish one basic figure that maintains its form during the whole process and can be observed at all four seasons.

In nature, there exist processes and objects that evidently are very similar to classical fractals but cannot be covered by the traditional approaches because several self-similarity mechanisms participate in the process of their construction. A new methodology (see [12]–[16]) allowing one not only to study traditional fractals but also to introduce and to investigate a new class of objects – *blinking fractals* – that are not covered by traditional theories studying self-similarity processes can be used for describing such processes. In this paper, the notion of blinking fractals is introduced and it is shown that they can be used to model season changes and processes of growth in biological systems. The paper not only proposes the first

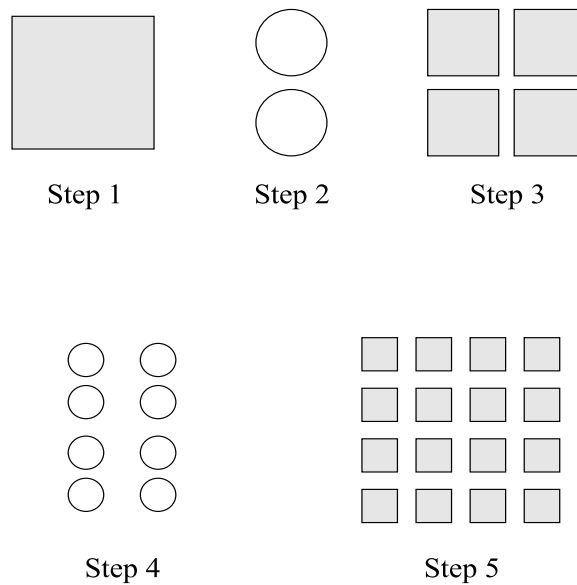


Figure 2: Results of the first five iterations.

example of such a modelling but also describes mathematical tools allowing one to study the properties of processes of growth in the limit even in the situations where various kinds of divergency take place.

The rest of the paper is organized as follows. Section 2 introduces the notion of blinking fractals, presents some examples, and formulates methodological postulates that will be used in the further investigation. Section 3 describes a new applied approach to infinity that allows one to study the blinking and traditional fractals at infinity and to execute arithmetical operations with infinite, finite, and infinitesimal numbers. In Section 4, it is shown how the blinking fractals can be investigated by using the introduced infinite and infinitesimal numbers. Section 5 shows how processes of growth of biological systems can be modelled by using the blinking fractals, particularly, the new applied approach to infinity is used to study a model of the growth of a forest. Finally, Section 6 concludes the paper.

## 2 Blinking fractals and a new methodology for studying infinite processes

Before going to a general definition of blinking fractals let us consider a process shown in Fig. 2. At the first moment we see a grey square with the side equal to 1. At the second moment we see two white circles with the diameter equal to  $\frac{1}{2}$ . Then each white circle is substituted by to grey squares  $\frac{1}{2}$  on side. This process of substitution continues in time as it shown in Fig. 2.

It is clear that the process shown in Fig. 2 is not a fractal process because at

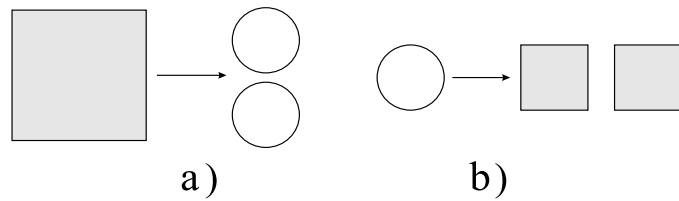


Figure 3: At each even iteration every square with a side equal to  $h$  is substituted by two circles having the diameter  $\frac{h}{2}$ . At each odd iteration every circle with a diameter  $d$  is substituted by two squares with  $d$  on side.

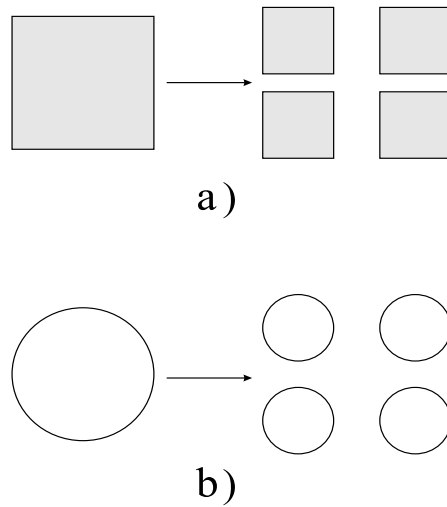


Figure 4: Two traditional fractal processes that can be extracted from the blinking fractal process shown in Fig. 2.

each even iteration squares are not transformed in smaller copies of themselves but in circles (see Fig. 2 and Fig. 3, left). Analogously, at odd iterations circles are transformed in squares instead of smaller circles (see Fig. 2 and Fig. 3, right). Thus, the process shown in Fig. 2 is a mixture of two fractal processes with the rules shown in Fig. 4: the first of them works with grey squares and the second with white circles.

Traditional approaches are not able to say anything about the behavior of this process from Fig. 2 at infinity. Does there exist a limit object of this process? If it exists, what can we say about its structure? Does it consist of white circles or grey squares and how many of them take part of this limit object? All these questions remain without answers if traditional mathematical tools are used for analysis of such processes.

In this paper, we give answers to these questions by using a new approach

developed in [12, 13, 14, 16] for dealing with infinite, finite, and infinitesimal numbers. The new methodology will be applied to study traditional fractals and new objects constructed using the principle of self-similarity with an infinite cyclic application of *several fractal rules*. These objects are called hereinafter *blinking fractals*.

Usually, when mathematicians deal with infinite objects (sets or processes) it is supposed that human beings are able to execute certain operations infinitely many times (see [1, 2, 8, 10]). For example, in a fixed numeral system it is possible to write down a numeral<sup>1</sup> with *any* number of digits. However, this supposition is an abstraction because we live in a finite world and all human beings and/or computers finish operations they have started.

The new computational paradigm introduced in [12, 13, 14, 16] does not use this abstraction and, therefore, is closer to the world of practical calculations than traditional approaches. Its strong computational character is enforced also by the fact that the first simulator of the Infinity Computer able to work with infinite, finite, and infinitesimal numbers introduced in [12, 13, 14, 16] has been already realized (see [7, 11]).

In order to introduce the new methodology, let us consider a study published in *Science* by Peter Gordon (see [5]) where he describes a primitive tribe living in Amazonia - Pirahã - that uses a very simple numeral system for counting: one, two, many. For Pirahã, all quantities bigger than two are just ‘many’ and such operations as  $2+2$  and  $2+1$  give the same result, i.e., ‘many’. Using their weak numeral system Pirahã are not able to see, for instance, numbers 3, 4, 5, and 6, to execute arithmetical operations with them, and, in general, to say anything about these numbers because in their language there are neither words nor concepts for that. Moreover, the weakness of their numeral system leads to such results as

$$\text{‘many’} + 1 = \text{‘many’}, \quad \text{‘many’} + 2 = \text{‘many’},$$

which are very familiar to us in the context of views on infinity used in the traditional calculus

$$\infty + 1 = \infty, \quad \infty + 2 = \infty.$$

This observation leads us to the following idea: *Probably our difficulty in working with infinity is not connected to the nature of infinity but is a result of inadequate numeral systems used to express numbers.*

We start by introducing three postulates that will fix our methodological positions with respect to infinite and infinitesimal quantities and mathematics, in general.

**Postulate 1.** *We accept that human beings and machines are able to execute only a finite number of operations.*

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<sup>1</sup>We remind that *numeral* is a symbol or group of symbols that represents a *number*. The difference between numerals and numbers is the same as the difference between words and the things they refer to. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols ‘7’, ‘seven’, and ‘VII’ are different numerals, but they all represent the same number.

Thus, we accept that we shall never be able to give a complete description of infinite processes and sets due to our finite capabilities. Particularly, this means that we accept that we are able to write down only a finite number of symbols to express numbers.

The second postulate that will be adopted is due to the following consideration. In natural sciences, researchers use tools to describe the object of their study and the used instrument influences results of observations. When physicists see a black dot in their microscope they cannot say: The object of observation *is* the black dot. They are obliged to say: the lens used in the microscope allows us to see the black dot and it is not possible to say anything more about the nature of the object of observation until we'll not change the instrument - the lens or the microscope itself - by a more precise one.

Due to Postulate 1, the same happens in mathematics studying natural phenomena, numbers, and objects that can be constructed by using numbers. Numeral systems used to express numbers are among the instruments of observations used by mathematicians. Usage of powerful numeral systems gives possibility to obtain more precise results in mathematics in the same way as usage of a good microscope gives a possibility to obtain more precise results in physics. However, the capabilities of all mathematical tools will be always limited due to Postulate 1.

**Postulate 2.** *Following natural sciences, we shall not tell **what are** the mathematical objects we deal with; we just shall construct more powerful tools that will allow us to improve our capacities to observe and to describe properties of mathematical objects.*

Particularly, this means that from our point of view, axiomatic systems do not define mathematical objects but just determine formal rules for operating with certain numerals reflecting some properties of the studied mathematical objects.

After all, we want to treat infinite and infinitesimal numbers in the same manner as we are used to deal with finite ones, i.e., by applying the philosophical principle of Ancient Greeks 'The part is less than the whole'. This principle, in our opinion, very well reflects organization of the world around us but is not incorporated in many traditional infinity theories where it is true only for finite numbers.

**Postulate 3.** *Following Ancient Greeks, we adopt the principle 'The part is less than the whole' to all numbers (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite).*

Due to this declared applied statement, such concepts as bijection, numerable and continuum sets, cardinal and ordinal numbers cannot be used in this paper because they belong to the theories working with different assumptions<sup>2</sup>. However, the approach proposed here does not contradict Cantor. In contrast, it evolves his deep ideas regarding existence of different infinite numbers in a more applied way.

Let us start our consideration by studying situations arising in practice when it is necessary to operate with extremely large quantities (see [12] for a detailed

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<sup>2</sup>As a consequence, the approach used in this paper is different also with respect to non-standard analysis introduced in [10] and built using Cantor's ideas.

discussion). Imagine that we are in a granary and the owner asks us to count how much grain he has inside it. There are a few possibilities of finding an answer to this question. The first one is to count the grain seed by seed. Of course, nobody can do this because the number of seeds is enormous.

To overcome this difficulty, people take sacks, fill them in with seeds, and count the number of sacks. It is important that nobody counts the number of seeds in a sack. At the end of the counting procedure, we shall have a number of sacks completely filled and some remaining seeds that are not sufficient to complete the next sack. At this moment it is possible to return to the seeds and to count the number of remaining seeds that have not been put in sacks (or a number of seeds that it is necessary to add to obtain the last completely full sack).

If the granary is huge and it becomes difficult to count the sacks, then trucks or even big train waggons are used. Of course, we suppose that all sacks contain the same number of seeds, all trucks – the same number of sacks, and all waggons – the same number of trucks. At the end of the counting we obtain a result in the following form: the granary contains 17 waggons, 23 trucks, 2 sacks, and 84 seeds of grain. Note, that if we add, for example, one seed to the granary, we can count it and see that the granary has more grain. If we take out one waggon, we again be able to say how much grain has been subtracted.

Thus, in our example it is necessary to count large quantities. They are finite but it is impossible to count them directly using elementary units of measure,  $u_0$ , i.e., seeds, because the quantities expressed in these units would be too large. Therefore, people are forced to behave as if the quantities were infinite.

To solve the problem of ‘infinite’ quantities, new units of measure,  $u_1, u_2$ , and  $u_3$ , are introduced (units  $u_1$  – sacks,  $u_2$  – trucks, and  $u_3$  – waggons). The new units have the following important peculiarity: it is not known how many units  $u_i$  there are in the unit  $u_{i+1}$  (we do not count how many seeds are in a sack, we just *complete* the sack). Every unit  $u_{i+1}$  is filled in completely by the units  $u_i$ . Thus, we know that all the units  $u_{i+1}$  contain a certain number  $K_i$  of units  $u_i$  but this number,  $K_i$ , is unknown. Naturally, it is supposed that  $K_i$  is the same for all instances of the units. Thus, numbers that it was impossible to express using only initial units of measure are perfectly expressible if new units are introduced. This key idea of counting by introduction of new units of measure will be used in the paper to deal with infinite quantities.

In order to have a possibility to write down infinite and infinitesimal numbers by a finite number of symbols, we need at least one new numeral expressing an infinite (or an infinitesimal) number corresponding to the chosen infinite unit of measure<sup>3</sup>. Then, it is necessary to propose a new numeral system fixing rules for writing down infinite and infinitesimal numerals and to describe arithmetical operations with them.

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<sup>3</sup>Note that introduction of a new numeral for expressing infinite and infinitesimal numbers is similar to introduction of the concept of zero and the numeral ‘0’ that in the past have allowed people to develop positional systems being more powerful than numeral systems existing before.

### 3 Infinite and infinitesimal numbers and operations with them

In this section, we give a brief introduction to the new applied approach to infinity (for a detailed consideration see [12]–[15]). Different numeral systems have been developed by humanity to describe finite numbers. More powerful numeral systems allow us to write down more numerals and, therefore, to express more numbers. A new positional numeral system with infinite radix described in this section evolves the idea of separate count of units with different exponents used in traditional positional systems to the case of infinite and infinitesimal numbers.

The infinite radix of the new system is introduced as the number of elements of the set  $\mathbb{N}$  of natural numbers expressed by the numeral  $\textcircled{1}$  called *grossone*. This mathematical object is introduced by describing its properties postulated by the *Infinite Unit Axiom* (IUA) consisting of three parts: Infinity, Identity, and Divisibility (we introduce them soon). This axiom is added to axioms for real numbers similarly to addition of the axiom determining zero to axioms of natural numbers when integer numbers are introduced. This means that it is postulated that associative and commutative properties of multiplication and addition, distributive property of multiplication over addition, existence of inverse elements with respect to addition and multiplication hold for grossone as for finite numbers.

Note that usage of a numeral indicating totality of the elements we deal with is not new in mathematics. It is sufficient to remind the theory of probability where events can be defined in two ways. First, as union of elementary events; second, as a sample space,  $\Omega$ , of all possible elementary events from where some elementary events have been excluded. Naturally, the second way to define events becomes particularly useful when the sample space consists of infinitely many elementary events.

The *Infinite Unit Axiom* consists of the following three statements:

*Infinity.* For any finite natural number  $n$  it follows  $n < \textcircled{1}$ .

*Identity.* The following relations link  $\textcircled{1}$  to identity elements 0 and 1

$$0 \cdot \textcircled{1} = \textcircled{1} \cdot 0 = 0, \quad \textcircled{1} - \textcircled{1} = 0, \quad \frac{\textcircled{1}}{\textcircled{1}} = 1, \quad \textcircled{1}^0 = 1, \quad 1^{\textcircled{1}} = 1, \quad 0^{\textcircled{1}} = 0. \quad (1)$$

*Divisibility.* For any finite natural number  $n$  sets  $\mathbb{N}_{k,n}, 1 \leq k \leq n$ , being the  $n$ th parts of the set,  $\mathbb{N}$ , of natural numbers have the same number of elements indicated by the numeral  $\frac{\textcircled{1}}{n}$  where

$$\mathbb{N}_{k,n} = \{k, k+n, k+2n, k+3n, \dots\}, \quad 1 \leq k \leq n, \quad \bigcup_{k=1}^n \mathbb{N}_{k,n} = \mathbb{N}. \quad (2)$$

Divisibility is based on Postulate 3. Let us illustrate it by three examples. If we take  $n = 1$ , then  $\mathbb{N}_{1,1} = \mathbb{N}$  and Divisibility tells that the set,  $\mathbb{N}$ , of natural numbers has  $\textcircled{1}$

elements. If  $n = 2$ , we have two sets  $\mathbb{N}_{1,2}$  and  $\mathbb{N}_{2,2}$  and they have  $\frac{\textcircled{1}}{2}$  elements each. If  $n = 3$ , then we have three sets  $\mathbb{N}_{1,3}$ ,  $\mathbb{N}_{2,3}$ , and  $\mathbb{N}_{3,3}$  having  $\frac{\textcircled{1}}{3}$  elements each.

$$\begin{array}{l} \textcircled{1} \rightarrow \mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \dots\} \\ \\ \frac{\textcircled{1}}{2} \begin{array}{l} \nearrow \mathbb{N}_{1,2} = \{1, 3, 5, 7, \dots\} \\ \searrow \mathbb{N}_{2,2} = \{2, 4, 6, \dots\} \end{array} \\ \\ \frac{\textcircled{1}}{3} \begin{array}{l} \nearrow \mathbb{N}_{1,3} = \{1, 4, 7, \dots\} \\ \rightarrow \mathbb{N}_{2,3} = \{2, 5, \dots\} \\ \searrow \mathbb{N}_{3,3} = \{3, 6, \dots\} \end{array} \end{array}$$

Before the introduction of the new positional system let us study some properties of grossone. First of all, as was already mentioned above, it is necessary to remind that  $\textcircled{1}$  is not either Cantor's  $\aleph_0$  or  $\omega$  that have been introduced in Cantor's theory on the basis of different assumptions. It will be shown hereinafter that grossone unifies both cardinal and ordinal aspects in the same way as finite numerals unify them. Its role in our infinite arithmetic is similar to the role of the number 1 in the finite arithmetic and it will serve us as the basis for construction of other infinite and infinitesimal numbers.

We start by the following important comment: to introduce  $\frac{\textcircled{1}}{n}$  we do not try to count elements  $k, k+n, k+2n, k+3n, \dots$ . In fact, we cannot do this due to the accepted Postulate 1. In contrast, we apply Postulate 3 and state that the number of elements of the  $n$ th part of the set, i.e.,  $\frac{\textcircled{1}}{n}$ , is  $n$  times less than the number of elements of the whole set, i.e., than  $\textcircled{1}$ . In terms of our granary example  $\textcircled{1}$  can be interpreted as the number of seeds in the sack. Then, if the sack contains  $\textcircled{1}$  seeds, its  $n$ th part contains  $\frac{\textcircled{1}}{n}$  seeds. It is worthy to emphasize that, since the numbers  $\frac{\textcircled{1}}{n}$  have been introduced as numbers of elements of sets  $\mathbb{N}_{k,n}$ , they are integer.

The introduced numerals  $\frac{\textcircled{1}}{n}$  and the sets  $\mathbb{N}_{k,n}$  allow us immediately to calculate the number of elements of certain infinite sets. For example, due to the introduced axiom, the set

$$\{3, 8, 13, 18, 23, 28, \dots\} = \mathbb{N}_{3,5}$$

and, therefore, has  $\frac{\textcircled{1}}{5}$  elements. The number of elements of sets being union, intersection, difference, or product of other sets of the type  $\mathbb{N}_{k,n}$  is defined in the same way as these operations are defined for finite sets. Thus, we can define the number of elements of sets being results of these operations with finite sets and infinite sets of the type  $\mathbb{N}_{k,n}$ . For example, the set

$$\{3, 8, 13, 18, 23, 28, \dots\} \setminus \{3, 23\} = \mathbb{N}_{3,5} \setminus \{3, 23\}$$

and, therefore, it has  $\frac{\textcircled{1}}{5} - 2$  elements.

Other results regarding calculating the number of elements of infinite sets can be found in [12, 16]. Particularly, it is shown that the number of elements of the set,  $\mathbb{Z}$ , of integers is equal to  $2\mathbb{1}$  and the number of elements of the set,  $\mathbb{Q}$ , of different rational numerals is equal to  $2\mathbb{1}^2$ .

The new numeral  $\mathbb{1}$  allows us to write down the set,  $\mathbb{N}$ , of natural numbers in the form

$$\mathbb{N} = \{1, 2, 3, \dots, \mathbb{1} - 2, \mathbb{1} - 1, \mathbb{1}\} \quad (3)$$

because *grossone has been introduced as the number of elements of the set of natural numbers* (similarly, the number 3 is the number of elements of the set  $\{1, 2, 3\}$ ). Thus, grossone is the biggest natural number and infinite numbers

$$\dots, \mathbb{1} - 3, \mathbb{1} - 2, \mathbb{1} - 1 \quad (4)$$

less than grossone are also natural numbers as the numbers  $1, 2, 3, \dots$ . They can be viewed both in terms of sets of numbers and in terms of grain. For example,  $\mathbb{1} - 1$  can be interpreted as the number of elements of the set  $\mathbb{N}$  from which a number has been excluded. In terms of our granary example  $\mathbb{1} - 1$  can be interpreted as a sack minus one seed.

Note that the set (3) is the same set of natural numbers we are used to deal with. Infinite numbers (4) also take part of the usual set,  $\mathbb{N}$ , of natural numbers<sup>4</sup>. The difficulty to accept existence of infinite natural numbers is in the fact that traditional numeral systems did not allow us to see them. In the same way as Pirahã are not able to see, for instance, numbers 3, 4, and 5 using their weak numeral system, traditional numeral systems did not allow us to see infinite natural numbers that we can see now using the new numeral  $\mathbb{1}$ .

Postulate 3 and the Infinite Unit Axiom allow us to obtain the following important result: the set  $\mathbb{N}$  is not a monoid under addition. In fact, the operation  $\mathbb{1} + 1$  gives us as the result a number greater than  $\mathbb{1}$ . Thus, by definition of grossone,  $\mathbb{1} + 1$  does not belong to  $\mathbb{N}$  and, therefore,  $\mathbb{N}$  is not closed under addition and is not a monoid.

This result also means that adding the Infinite Unit Axiom to the axioms of natural numbers defines the set of *extended natural numbers* indicated as  $\widehat{\mathbb{N}}$  and including  $\mathbb{N}$  as a proper subset

$$\widehat{\mathbb{N}} = \{1, 2, \dots, \mathbb{1} - 1, \mathbb{1}, \mathbb{1} + 1, \dots, \mathbb{1}^2 - 1, \mathbb{1}^2, \mathbb{1}^2 + 1, \dots\}. \quad (5)$$

Again, extended natural numbers greater than grossone can also be interpreted in the terms of sets of numbers. For example,  $\mathbb{1} + 3$  as the number of elements of the set  $\mathbb{N} \cup \{a, b, c\}$  where numbers  $a, b, c \notin \mathbb{N}$  and  $\mathbb{1}^2$  as the number of elements of the set

$$C = \{(a_1, a_2) : a_1 \in \mathbb{N}, a_2 \in \mathbb{N}\}.$$

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<sup>4</sup>This point is one of the differences with respect to non-standard analysis (see [8, 10]) where infinite numbers are not included in  $\mathbb{N}$ .

In terms of our granary example  $\textcircled{1} + 3$  can be interpreted as one sack plus three seeds and  $\textcircled{1}^2$  as a truck if we accept that the numbers  $K_i$  from page 7 are such that  $K_1 = K_2 = \textcircled{1}$ .

Analogously, we can consider the set,  $\widehat{\mathbb{Z}}$ , of *extended integer numbers*

$$\widehat{\mathbb{Z}} = \{\dots, -\textcircled{1} - 1, -\textcircled{1}, -\textcircled{1} + 1, \dots, -2, -1, 0, 1, 2, \dots, \textcircled{1} - 1, \textcircled{1}, \textcircled{1} + 1, \dots\}.$$

What can we say now about the number of elements of the sets  $\widehat{\mathbb{N}}$  and  $\widehat{\mathbb{Z}}$ ? Our positional numeral system with the radix  $\textcircled{1}$  does not allow us to say anything because it does not contain numerals able to express such numbers (see Postulates 1 and 2). It is necessary to introduce in a way a more powerful numeral system defining new numerals  $\textcircled{2}$ ,  $\textcircled{3}$ , etc. However, in spite of the fact that the numeral system using grossone does not allow us to express the numbers of elements of  $\widehat{\mathbb{N}}$  and  $\widehat{\mathbb{Z}}$ , we can work with those subsets of  $\widehat{\mathbb{N}}$  and  $\widehat{\mathbb{Z}}$  that can be defined by using numerals written down in our positional numeral system with the radix  $\textcircled{1}$ .

We have already started to write down simple infinite numbers and to execute arithmetical operations with them without concentrating our attention upon this question. In general, to express a number  $C$  in the new numeral positional system with base  $\textcircled{1}$  we subdivide  $C$  into groups corresponding to powers of  $\textcircled{1}$ :

$$C = c_{p_m} \textcircled{1}^{p_m} + \dots + c_{p_1} \textcircled{1}^{p_1} + c_{p_0} \textcircled{1}^{p_0} + c_{p_{-1}} \textcircled{1}^{p_{-1}} + \dots + c_{p_{-k}} \textcircled{1}^{p_{-k}}. \quad (6)$$

Then, the record

$$C = c_{p_m} \textcircled{1}^{p_m} \dots c_{p_1} \textcircled{1}^{p_1} c_{p_0} \textcircled{1}^{p_0} c_{p_{-1}} \textcircled{1}^{p_{-1}} \dots c_{p_{-k}} \textcircled{1}^{p_{-k}} \quad (7)$$

represents the number  $C$ , where finite numbers  $c_i$  are called *infinite grossdigits* and can be both positive and negative; numbers  $p_i$  are called *grosspowers* and can be finite, infinite, and infinitesimal (the introduction of infinitesimal numbers will be given soon). The numbers  $p_i$  are such that  $p_i > 0, p_0 = 0, p_{-i} < 0$  and

$$p_m > p_{m-1} > \dots p_2 > p_1 > p_{-1} > p_{-2} > \dots p_{-(k-1)} > p_{-k}.$$

In the traditional positional systems there exists a convention that a digit  $a_i$  shows how many powers  $b^i$  are present in the number and the radix  $b$  is not written explicitly. In the record (7) we write  $\textcircled{1}^{p_i}$  explicitly because in the new numeral positional system the number  $i$  in general is not equal to the grosspower  $p_i$ . This gives possibility to write, for example, such numbers as  $\frac{7}{3} \textcircled{1}^4 \frac{84}{19} \textcircled{1}^{-3.1}$  where  $p_1 = 4, p_{-1} = -3.1$ . Grossdigits  $c_i, -k \leq i \leq m$ , can be integer or fractional and expressed by many symbols; in our example,  $c_4 = \frac{7}{3}$  and  $c_{-3.1} = \frac{84}{19}$ .

*Finite numbers* in this new numeral system are represented by numerals having only one grosspower equal to zero. In fact, if we have a number  $C$  such that  $m = k = 0$  in representation (7), then due to (1) we have  $C = c_0 \textcircled{1}^0 = c_0$ . Thus, the number  $C$  in this case does not contain infinite units and is equal to the grossdigit  $c_0$  which being a conventional finite number can be expressed by any positional system with finite base  $b$  (or by another traditional numeral system). It is important

to emphasize that the grossdigit  $c_0$  can be integer or fractional and can be expressed by a few symbols in contrast to the traditional positional systems where each digit is integer and is represented by one symbol from the alphabet  $\{0, 1, 2, \dots, b-1\}$ . Thus, the grossdigit  $c_0$  shows how many finite units and/or parts of the finite unit,  $1 = \mathbb{1}^0$ , there are in the number  $C$ .

*Infinite numbers* in this numeral system are expressed by numerals having at least one positive finite or infinite grosspower. In the following example the left-hand expression presents the way to write down infinite numbers and the right-hand shows how the value of the number is calculated:

$$21.4\mathbb{1}^{23} - 1.45\mathbb{1}^{3.4} 852.1\mathbb{1}^{-66.2} = 21.4\mathbb{1}^{23} - 1.45\mathbb{1}^{3.4} + 852.1\mathbb{1}^{-66.2}.$$

If a grossdigit  $c_{p_i}$  is equal to 1 then we write  $\mathbb{1}^{p_i}$  instead of  $1\mathbb{1}^{p_i}$ . Analogously, if power  $\mathbb{1}^0$  is the lowest in a number then we often use simply the corresponding grossdigit  $c_0$  without  $\mathbb{1}^0$ , for instance, we write  $23\mathbb{1}^{14}5$  instead of  $23\mathbb{1}^{14}5\mathbb{1}^0$  or 8 instead of  $8\mathbb{1}^0$ .

*Infinitesimal numbers* are represented by numerals having only negative finite or infinite grosspowers. The simplest number from this group is  $\mathbb{1}^{-1} = \frac{1}{\mathbb{1}}$  being the inverse element with respect to multiplication for  $\mathbb{1}$ :

$$\frac{1}{\mathbb{1}} \cdot \mathbb{1} = \mathbb{1} \cdot \frac{1}{\mathbb{1}} = 1. \quad (8)$$

Note that all infinitesimals are not equal to zero. Particularly,  $\frac{1}{\mathbb{1}} > 0$  because  $1 > 0$  and  $\mathbb{1} > 0$ . It has a clear interpretation in our granary example. Namely, if we have a sack and it contains  $\mathbb{1}$  seeds then one sack divided by  $\mathbb{1}$  is equal to one seed. Vice versa, one seed, i.e.,  $\frac{1}{\mathbb{1}}$ , multiplied by the number of seeds in the sack,  $\mathbb{1}$ , gives one sack of seeds.

Let us now introduce arithmetical operations for infinite, infinitesimal, and finite numbers (see [12] for a detailed discussion and examples). The numbers  $A$ ,  $B$ , and their sum  $C$  are represented in the record of the type

$$A = \sum_{i=1}^K a_{k_i} \mathbb{1}^{k_i}, \quad B = \sum_{j=1}^M b_{m_j} \mathbb{1}^{m_j}, \quad C = \sum_{i=1}^L c_{l_i} \mathbb{1}^{l_i}. \quad (9)$$

The operation of *addition* of two given infinite numbers  $A$  and  $B$  returns as the result an infinite number  $C$  constructed as follows (the operation of subtraction is a direct consequence of that of addition and is thus omitted). Then the result  $C$  is constructed by including in it all items  $a_{k_i} \mathbb{1}^{k_i}$  from  $A$  such that  $k_i \neq m_j$ ,  $1 \leq j \leq M$ , and all items  $b_{m_j} \mathbb{1}^{m_j}$  from  $B$  such that  $m_j \neq k_i$ ,  $1 \leq i \leq K$ . If in  $A$  and  $B$  there are items such that  $k_i = m_j$  for some  $i$  and  $j$  then this grosspower  $k_i$  is included in  $C$  with the grossdigit  $b_{k_i} + a_{k_i}$ , i.e., as  $(b_{k_i} + a_{k_i}) \mathbb{1}^{k_i}$ . It can be seen from this definition that the introduced operation enjoys the usual properties of commutativity and associativity due to definition of grossdigits and the fact that addition for each grosspower of  $\mathbb{1}$  is executed separately.

The operation of *multiplication* of two given infinite numbers  $A$  and  $B$  from (9) returns as the result the infinite number  $C$  constructed as follows.

$$C = \sum_{j=1}^M C_j, \quad C_j = b_{m_j} \mathbb{1}^{m_j} \cdot A = \sum_{i=1}^K a_{k_i} b_{m_j} \mathbb{1}^{k_i+m_j}, \quad 1 \leq j \leq M. \quad (10)$$

Similarly to addition, the introduced multiplication is commutative and associative. It is easy to show that the distributive property is also valid for these operations.

In the operation of *division* of a given infinite number  $C$  by an infinite number  $B$  we obtain an infinite number  $A$  and a remainder  $R$  that can be also equal to zero, i.e.,  $C = A \cdot B + R$ .

The number  $A$  is constructed as follows. The numbers  $B$  and  $C$  are represented in the form (9). The first grossdigit  $a_{k_K}$  and the corresponding maximal exponent  $k_K$  are established from the equalities

$$a_{k_K} = c_{l_L} / b_{m_M}, \quad k_K = l_L - m_M. \quad (11)$$

Then the first partial remainder  $R_1$  is calculated as

$$R_1 = C - a_{k_K} \mathbb{1}^{k_K} \cdot B. \quad (12)$$

If  $R_1 \neq 0$  then the number  $C$  is substituted by  $R_1$  and the process is repeated by a complete analogy. The grossdigit  $a_{k_{K-i}}$ , the corresponding grosspower  $k_{K-i}$  and the partial remainder  $R_{i+1}$  are computed by formulae (13) and (14) obtained from (11) and (12) as follows:  $l_L$  and  $c_{l_L}$  are substituted by the highest grosspower  $n_i$  and the corresponding grossdigit  $r_{n_i}$  of the partial remainder  $R_i$  that in its turn substitutes  $C$ :

$$a_{k_{K-i}} = r_{n_i} / b_{m_M}, \quad k_{K-i} = n_i - m_M. \quad (13)$$

$$R_{i+1} = R_i - a_{k_{K-i}} \mathbb{1}^{k_{K-i}} \cdot B, \quad i \geq 1. \quad (14)$$

The process stops when a partial remainder equal to zero is found (this means that the final remainder  $R = 0$ ) or when a required accuracy of the result is reached.

## 4 Numerical analysis of blinking fractals

In the previous section, new numerals expressing infinite and infinitesimal numbers have been introduced. These numerals give us a possibility to study traditional fractals and blinking fractals at different points of infinity and to use them for modelling the nature. In this section, we investigate the blinking fractal introduced in Section 2 and infinite sequences will be used for this goal. Naturally, we need first to understand what can we say about the infinite sequences using the new mathematical language.

We start by reminding traditional definitions of the infinite sequences and sub-sequences. An *infinite sequence*  $\{a_n\}, a_n \in A, n \in \mathbb{N}$ , is a function having as the domain the set of natural numbers,  $\mathbb{N}$ , and as the codomain a set  $A$ . A *subsequence* is a sequence from which some of its elements have been cancelled. The IUA allows us to prove the following result.

**Theorem 1.** *The number of elements of any infinite sequence is less or equal to  $\textcircled{1}$ .*

*Proof.* The IUA states that the set  $\mathbb{N}$  has  $\textcircled{1}$  elements. Thus, due to the sequence definition given above, any sequence having  $\mathbb{N}$  as the domain has  $\textcircled{1}$  elements.

The notion of subsequence is introduced as a sequence from which some of its elements have been cancelled. Thus, this definition gives infinite sequences having the number of members less than grossone.  $\square$

It becomes appropriate now to define the *complete sequence* as an infinite sequence containing  $\textcircled{1}$  elements. For example, the sequence  $\{n\}$  of natural numbers is complete, the sequences of even and odd natural numbers are not complete.

One of the immediate consequences of the understanding of this result is that any sequential process can have at maximum  $\textcircled{1}$  elements and, due to Postulate 1, it depends on the chosen numeral system which numbers among  $\textcircled{1}$  members of the process we can observe.

*Example 4.1.* Let us consider the set,  $\widehat{\mathbb{N}}$ , of extended natural numbers from (5). Then, starting from the number 1, the process of the sequential counting can arrive at maximum to  $\textcircled{1}$

$$\underbrace{1, 2, 3, 4, \dots, \textcircled{1}-2, \textcircled{1}-1, \textcircled{1}, \textcircled{1}+1, \textcircled{1}+2, \textcircled{1}+3, \dots}_{\textcircled{1}}$$

Starting from 2 it arrives at maximum to  $\textcircled{1}+1$

$$\underbrace{1, 2, 3, 4, \dots, \textcircled{1}-2, \textcircled{1}-1, \textcircled{1}, \textcircled{1}+1, \textcircled{1}+2, \textcircled{1}+3, \dots}_{\textcircled{1}}$$

Starting from 3 it arrives at maximum to  $\textcircled{1}+2$

$$\underbrace{1, 2, 3, 4, \dots, \textcircled{1}-2, \textcircled{1}-1, \textcircled{1}, \textcircled{1}+1, \textcircled{1}+2, \textcircled{1}+3, \dots}_{\textcircled{1}} \quad \square$$

Similarly to infinite sets, the IUA imposes a more precise description of infinite sequences. To define a sequence  $\{a_n\}$  it is not sufficient just to give a formula for  $a_n$ . It is necessary to indicate explicitly its number of elements.

*Example 4.2.* Let us consider the following three sequences,  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$ :

$$\{a_n\} = \{2, 4, \dots, 2(\textcircled{1}-1), 2\textcircled{1}\},$$

$$\{b_n\} = \{2, 4, \dots, 2\left(\frac{2\textcircled{1}}{5}-1\right), 2 \cdot \frac{2\textcircled{1}}{5}\}, \quad (15)$$

$$\{c_n\} = \{2, 4, \dots, 2\left(\frac{4\textcircled{1}}{5}-1\right), 2 \cdot \frac{4\textcircled{1}}{5}\}. \quad (16)$$

They have the same general element equal to  $2n$  but they are different because they have different number of members. The first sequence has  $\textcircled{1}$  elements and is thus complete, the other two sequences are not complete:  $\{b_n\}$  has  $\frac{2\textcircled{1}}{5}$  elements and

$\{c_n\}$  has  $\frac{4\textcircled{1}}{5}$  members. Note also that among these three sequences only  $\{b_n\}$  is a subsequence of the sequence of even natural numbers because its last element has the number  $\frac{2\textcircled{1}}{5} \leq \frac{\textcircled{1}}{2}$ . Since  $\textcircled{1}$  is the last even natural number, elements of  $\{a_n\}$  and  $\{c_n\}$  having  $n > \frac{\textcircled{1}}{2}$  are not natural but extended natural numbers (see (5)).  $\square$

Thus, to describe a sequence we should use the record  $\{a_n : k\}$  where  $a_n$  is, as usual, the general element and  $k$  is the number (finite or infinite) of members of the sequence. In connection to this definition the following natural question arises inevitably. Suppose that we have two sequences, for example,  $\{b_n : \frac{2\textcircled{1}}{5}\}$  and  $\{c_n : \frac{4\textcircled{1}}{5}\}$  from (15) and (16). Can we create a new sequence,  $\{d_n : k\}$ , composed from both of them, for instance, as it is shown below

$$b_1, b_2, \dots, b_{\frac{2\textcircled{1}}{5}-2}, b_{\frac{2\textcircled{1}}{5}-1}, b_{\frac{2\textcircled{1}}{5}}, c_1, c_2, \dots, c_{\frac{4\textcircled{1}}{5}-2}, c_{\frac{4\textcircled{1}}{5}-1}, c_{\frac{4\textcircled{1}}{5}}$$

and which will be the value of the number of its elements  $k$ ?

The answer is ‘no’ because due to the definition of the infinite sequence, a sequence can be at maximum complete, i.e., it cannot have more than  $\textcircled{1}$  elements. Starting from the element  $b_1$  we can arrive at maximum to the element  $c_{\frac{3\textcircled{1}}{5}}$  being the element number  $\textcircled{1}$  in the sequence  $\{d_n : k\}$  which we try to construct. Therefore,  $k = \textcircled{1}$  and

$$\underbrace{b_1, \dots, b_{\frac{2\textcircled{1}}{5}}, c_1, \dots, c_{\frac{3\textcircled{1}}{5}}}_{\textcircled{1} \text{ elements}}, \underbrace{c_{\frac{3\textcircled{1}}{5}+1}, \dots, c_{\frac{4\textcircled{1}}{5}}}_{\frac{\textcircled{1}}{5} \text{ elements}}$$

The remaining members of the sequence  $\{c_n : \frac{4\textcircled{1}}{5}\}$  will form the second sequence,  $\{g_n : l\}$  having  $l = \frac{4\textcircled{1}}{5} - \frac{3\textcircled{1}}{5} = \frac{\textcircled{1}}{5}$  elements. Thus, we have formed two sequences, the first of them is complete and the second is not.

The introduced more precise description of sequences allows us to observe fractal processes at different points of infinity by indicating the number of a step,  $n, 1 \leq n \leq \textcircled{1}$ , that we want to study. For example, for our blinking fractal from Fig. 2 we are able to say that we observe grey squares at all odd steps and white circles at even steps independently of the fact  $n$  is finite or infinite. Particularly, since due to the Infinite Unit Axiom  $\frac{\textcircled{1}}{2}$  is integer, for  $n = \textcircled{1}$  we have white circles and for  $n = \textcircled{1} - 1$  – grey squares.

In order to be able to measure fractals at infinity (e.g., to calculate the number of squares or circles at a step  $n$  in our blinking fractal from Fig. 2), we should reconsider the theory of divergent series from the new viewpoint introduced in the previous sections. The introduced numeral system allows us to express not only different finite numbers but also different infinite numbers. Therefore, due to Postulate 3, we should explicitly indicate the number of items in all sums independently on the fact whether this number is finite or infinite. Due to Postulate 2, we shall be able to calculate the sum if its items, the number of items, and the result are expressible in the numeral system used for calculations. It is important to notice that even though a sequence cannot have more than  $\textcircled{1}$  elements, the number

of items in a sum can be greater than grossone because the process of summing up not necessary should be executed by a sequential addition of items.

For instance, let us consider two infinite series

$$S_1 = 1 + 2 + 4 + 8 + 16 + \dots, \quad S_2 = 1 + 2 + 1 + 2 + 1 + 2 + 1 \dots$$

The traditional analysis gives us a very poor answer that both of them diverge to infinity. Such operations as  $S_1 - S_2$  or  $\frac{S_1}{S_2}$  are not defined. From the new point of view, the sums  $S_1$  and  $S_2$  can be calculated but, due to Postulate 3, it is necessary to indicate explicitly the number of items in both sums.

Suppose that the sum  $S_1$  has  $m + 1$  items and the sum  $S_2$  has  $n$  items:

$$S_1(m) = \underbrace{1 + 2 + 4 + 8 + \dots + 2^m}_{m+1}, \quad S_2(n) = \underbrace{1 + 2 + 1 + 2 + 1 + \dots}_n \quad (17)$$

Let us first calculate the sum  $S_1(m)$ . It is evident that it is a particular case of the sum

$$Q_m = \sum_{i=0}^m q^i = 1 + q + q^2 + \dots + q^m, \quad (18)$$

where  $m$  can be finite or infinite. Traditional analysis proves that the geometric series  $\sum_{i=0}^{\infty} q^i$  converges to  $\frac{1}{1-q}$  for  $q$  such that  $-1 < q < 1$ . We are able to give a more precise answer for *all* values of  $q$  and finite and infinite values of  $m$ .

By multiplying the left hand and the right hand parts of this equality by  $q$  and by subtracting the result from (18) we obtain

$$Q_m - qQ_m = 1 - q^{m+1}$$

and, as a consequence, for all  $q \neq 1$  the formula

$$Q_m = \frac{1 - q^{m+1}}{1 - q} \quad (19)$$

holds for finite and infinite  $m$ . Thus, the possibility to express infinite and infinitesimal numbers allows us to take into account infinite  $m$  too and the value  $q^{m+1}$  being infinitesimal for a finite  $q < 1$  and infinite for  $q > 1$ . Moreover, we can calculate  $Q_m$  for  $q = 1$  also because in this case we have just

$$Q_m = \underbrace{1 + 1 + 1 + \dots + 1}_{m+1} = m + 1.$$

Now, to calculate the sum  $S_1(m)$  it is sufficient to take  $q = 2$

$$S_1(m) = \underbrace{1 + 2 + 4 + 8 + 16 + \dots + 2^m}_{m+1} = \frac{1 - 2^{m+1}}{1 - 2} = 2^{m+1} - 1.$$

This formula can be used for finite and infinite values of  $m$ . For instance, if  $m = \frac{\textcircled{1}}{2} - 1$  then  $S_1(\frac{\textcircled{1}}{2} - 1) = 2^{\frac{\textcircled{1}}{2}} - 1$ ; if  $m = \frac{\textcircled{1}}{2}$  then  $S_1(\frac{\textcircled{1}}{2}) = 2^{\frac{\textcircled{1}}{2}+1} - 1$ . Note that

the sum  $S_1(\frac{\textcircled{1}}{2})$  has been obtained by adding  $2^{\frac{\textcircled{1}}{2}}$  to the sum  $S_1(\frac{\textcircled{1}}{2} - 1)$ . In fact, if we subtract from the obtained number  $2^{\frac{\textcircled{1}}{2}+1} - 1$  this value, we obtain exactly  $S_1(\frac{\textcircled{1}}{2} - 1)$ :

$$S_1(\frac{\textcircled{1}}{2}) - 2^{\frac{\textcircled{1}}{2}} = 2^{\frac{\textcircled{1}}{2}+1} - 1 - 2^{\frac{\textcircled{1}}{2}} = 2^{\frac{\textcircled{1}}{2}} - 1 = S_1(\frac{\textcircled{1}}{2} - 1).$$

The second sum,  $S_2(n)$ , from (17) is calculated as follows

$$S_2(n) = \underbrace{1+2+1+2+1+\dots}_n = \begin{cases} k+2k = 3k, & \text{if } n = 2k, \\ k+2k+1 = 3k+1, & \text{if } n = 2k+1. \end{cases}$$

By giving numerical values (finite or infinite) to  $n$  we obtain numerical values for results of the sum. If, for instance,  $n = 3\textcircled{1}$  then we obtain  $S_2(3\textcircled{1}) = 4.5\textcircled{1}$  because  $\textcircled{1}$  is even. If  $n = 3\textcircled{1} + 1$  then we obtain  $S_2(3\textcircled{1}) = 4.5\textcircled{1} + 1$ . Note, that we have no indeterminate expressions and the results of  $S_1(m) - S_2(n)$  and  $\frac{S_1(m)}{S_2(n)}$  can be easily calculated.

Let us now return to fractals. First of all, it is evident that the number of circles or squares at a step  $i, 1 \leq i \leq \textcircled{1}$ , in the blinking process from Fig. 2) is defined by the sum  $S_1(i - 1)$ . It is important to remind that, due to the IUA, a process cannot have more than grossone steps but a sum can have more than grossone items because it can be calculated in parallel (it is important that it is not calculated in sequence). Thus, if we consider the process from Fig. 2, then in  $S_1(i)$  it follows  $1 \leq i \leq \textcircled{1}$  but, of course, it is possible to calculate  $S_1(i), i \geq \textcircled{1}$ , if this is done without any connection to processes or in connection with another process (for instance, starting the process from two circles instead of one square, it is possible to arrive to  $S_1(\textcircled{1})$ , see discussion in example 4.1).

We conclude this section by calculating the side of the squares,  $s(i), i = 2k - 1, 1 \leq k \leq \frac{\textcircled{1}}{2}$ , and the diameter of circles,  $d(i), i = 2k, 1 \leq k \leq \frac{\textcircled{1}}{2}$ , for the blinking fractal from Fig. 2. It is easily to show that

$$s(i) = \frac{1}{2^{k-1}}, \quad i = 2k - 1, \quad 1 \leq k \leq \frac{\textcircled{1}}{2},$$

$$d(i) = \frac{1}{2^k}, \quad i = 2k, \quad 1 \leq k \leq \frac{\textcircled{1}}{2}.$$

For finite values of  $i$  we obtain finite values of  $s(i)$  and  $d(i)$ , whereas for infinite values of  $i$  we obtain infinitesimal values of  $s(i)$  and  $d(i)$ . For example, for  $i = \frac{\textcircled{1}}{3}$  it follows that we have white circles (because due to the IUA, for all finite integer  $n$  the numbers of the form  $\frac{\textcircled{1}}{n}$  are integer and, therefore,  $i = \frac{\textcircled{1}}{3}$  is even) and their diameter  $d(\frac{\textcircled{1}}{3}) = 2^{-\frac{\textcircled{1}}{6}}$ . Analogously, for  $i = \frac{\textcircled{1}}{3} - 1$  we have grey squares having the side  $s(\frac{\textcircled{1}}{3} - 1) = 2^{1-\frac{\textcircled{1}}{6}}$ .

Thus, the new infinite and infinitesimal numerals allow us to observe and to measure the traditional and blinking fractals at different points of infinity.

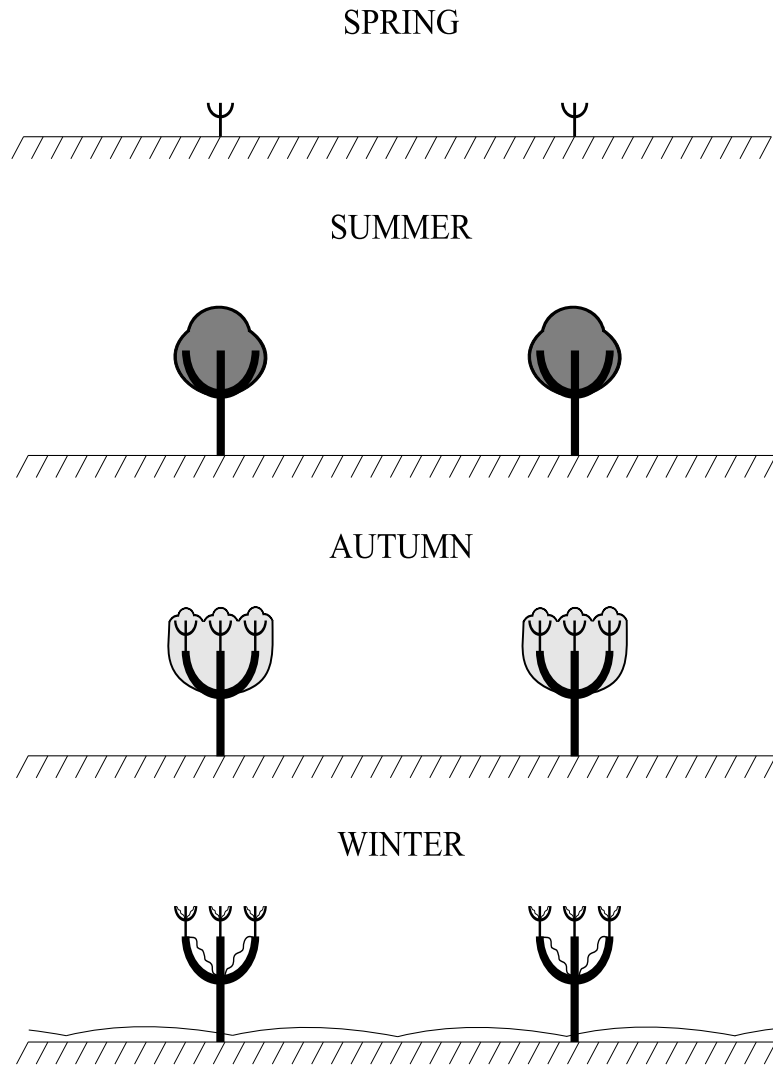


Figure 5: The first year of growth.

## 5 A blinking fractals model of the growth of a forest and its analysis at infinity

The analysis performed in the previous section allows us to pass to modelling processes in nature having a blinking fractals structure. Without loss of the generality we consider biennial plants (called hereinafter for simplicity *trees*) that will be observed four times a year: in spring, in summer, in autumn, and in winter. The process of growth starts in spring by planting two small trees (see Fig. 5, spring) in a line. In summer, the trees grow up and green leaves (shown by grey color) appear. During summer new branches appear and when we observe our trees in autumn, we see these new branches and leaves that meanwhile have become yellow (shown

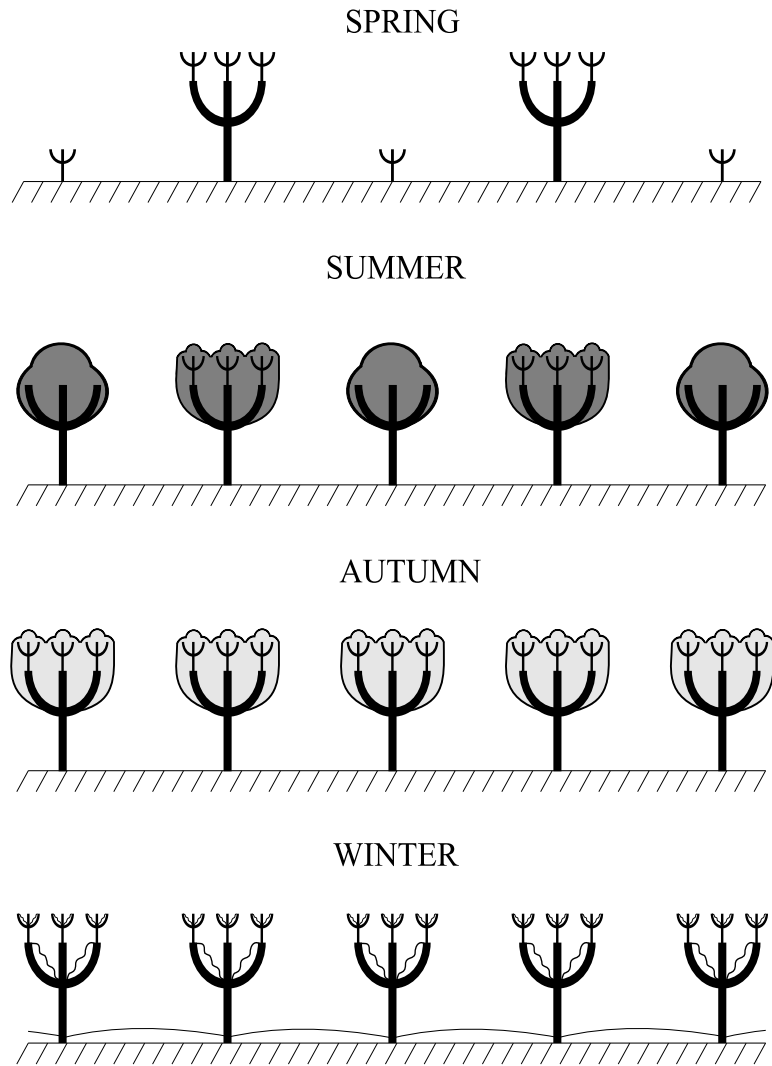


Figure 6: The second year of growth.

in Fig. 5 by the light grey color). When we observe our small forest in winter, there are no leaves and the trees are under the snow.

When we observe our forest in spring of the second year (see Fig. 6, spring), we see that three new trees have appeared (we suppose that the growth goes along the line defined by the first two trees). In summer of the second year, we see green leaves and observe that the new trees have grown up but the old two trees are not able to grow up and remain the same. In autumn, all five trees have the same measure and yellow leaves. In winter, all of them are under the snow. During the winter two old trees die and at their places new young trees appear. Two more new trees appear also at the free places on the left and the right ends of our forest. Thus, in spring of the third year we observe the situation shown in Fig. 7.

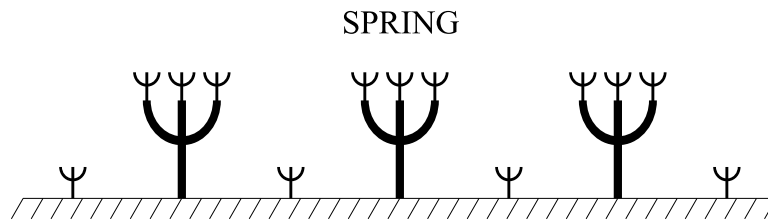


Figure 7: Spring of the third year of growth.

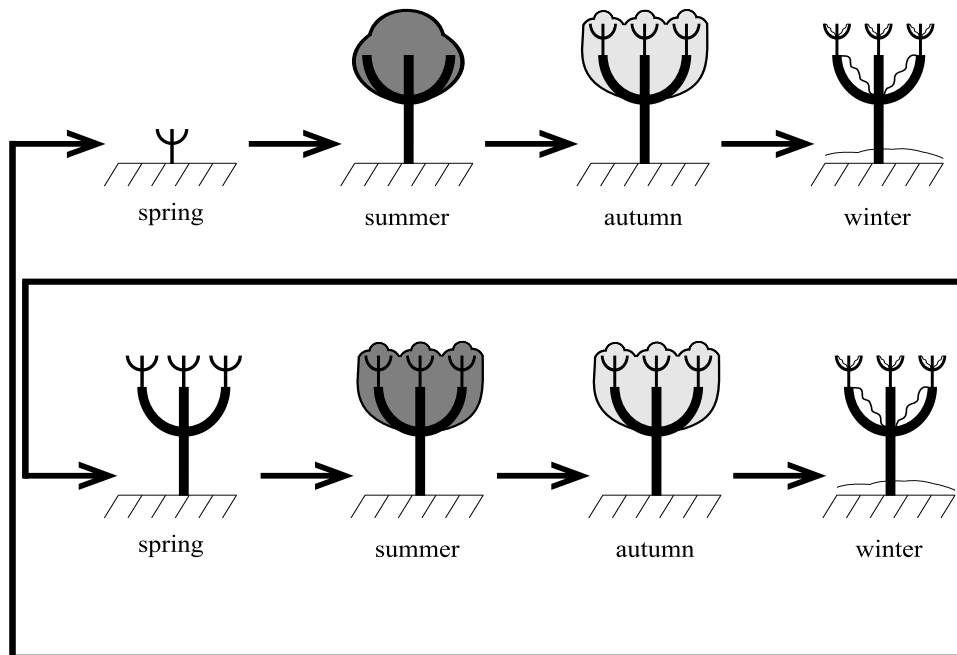


Figure 8: The two years cycle of growth.

The process then continues to infinity and at each place where a tree appears we can observe the two years cycle shown in Fig. 8. It is evident that the described model is not a fractal. By using traditional mathematics we are not able to answer to the following questions: How many trees and how many branches will have our forest at infinity? What will be color of the leaves? However, we can separate from the process of growth several processes behaving as fractals and, as a consequence, the process of growth of our forest is a blinking fractal. The new approach introduced in the previous section will allow us to give quantitative answers to the questions above easily.

It is evident that during each season we have different basic figures and it is necessary to consider the forest at each season separately by applying the methodology of blinking fractals. Let us start from winter. If  $n$  is the number of the year

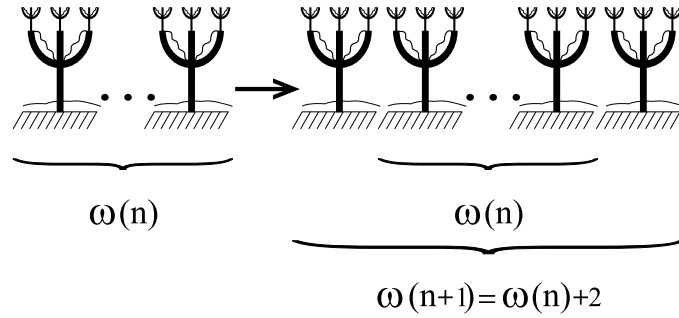


Figure 9: The winter process.

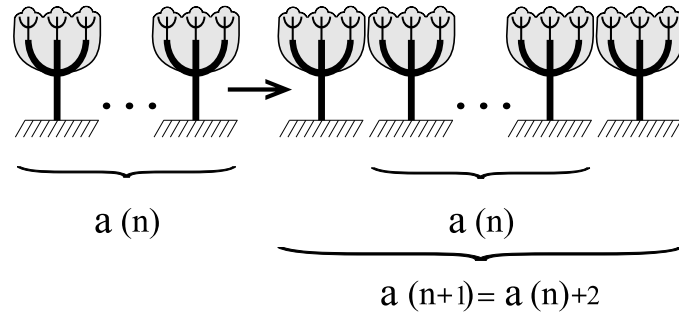


Figure 10: The autumn process.

then (see Fig. 9) we can easily calculate the number of trees at the forest,  $\omega(n)$ , for each year  $n$  as follows

$$\omega(1) = 2, \quad \omega(2) = 5, \quad \omega(3) = 7, \dots \quad \omega(n) = 2n + 1, \quad n \geq 2.$$

An analogous formula for calculating the number of the trees in autumn,  $a(n)$ , can be obtained (see Fig. 10) for the autumn process

$$a(1) = 2, \quad a(2) = 5, \quad a(3) = 7, \dots \quad a(n) = 2n + 1, \quad n \geq 2.$$

The same formulae for calculating the number of trees can be obtained for spring and summer, because the number of trees is the same during four seasons of each fixed year. Note that since we observe our forest four time per year and any process, due to the IUA, cannot have more than ① elements, the following restriction exists for the number,  $n$ , of the years of observations of our forest:  $1 \leq n \leq \frac{\textcircled{1}}{4}$ . This means, particularly, that the number of the trees at the last year  $a(\frac{\textcircled{1}}{4}) = \frac{\textcircled{1}}{2} + 1$ .

Although the number of the trees does not change during each fixed year, the changes take place for the number of branches; the form of the basic figures is also different for each season. Moreover, as it emphasized in Figs. 11 and 12, the processes in summer and spring are more complex than the processes in winter



$i = 3 + 4(n - 1)$  correspond to autumn at the year  $n$  and, consequently, the number of branches in autumn  $b(3 + 4(n - 1)) = 12(2n + 1)$  as well.

In spring and summer, the situation is different: young and old trees have different number of branches (see Figs. 5,6,11, and 12). Let us consider summer (spring can be studied by a complete analogy). In summer young trees have three big branches each and old trees have 12 branches each: three big and nine small.

Remind, that the observations at the year  $n$  corresponding to summer have numbers  $i = 2 + 4(n - 1)$ . Thus, in order to obtain the required result it is sufficient to use formulae (20) and (21) that give us

$$b(2) = 6, \quad b(2 + 4(n - 1)) = 3(n + 1) + 9n = 12n + 3, \quad 2 \leq n \leq \frac{\textcircled{1}}{4}.$$

For example, in summer of the last possible year of observation,  $n = \frac{\textcircled{1}}{4}$ , our forest has  $3\textcircled{1} + 3$  branches.

## 6 Concluding remarks

Fractals have been widely used in literature to model complex systems. In this paper, a new type of objects – blinking fractals – that are not covered by traditional theories studying self-similarity processes have been introduced for studying season changes during the growth of biological systems. The blinking fractals have been investigated together with traditional fractals using infinite and infinitesimal numbers proposed recently. It has been shown that the new approach allows one to obtain different quantitative characteristics of the behavior of both types of fractals.

As an example of application of the developed mathematical tools for describing the behavior of complex biological systems a new model of growth of a forest has been introduced and investigated using the notion of the blinking fractals. The new mathematical tools introduced in the paper have allowed us to separate in this complex model several fractal processes and to perform their accurate quantitative analysis. It is evident that the introduced model can be easily generalized to describe more complex objects and systems. For example, it is possible to introduce plants with the cycle of life superior to two years, the plants having a more complex structure can be also described by the introduced approach. It is possible also to introduce some additional mechanisms (for instance, plant pests or nature disasters) that influence the process of growth. These generalizations will be studied in the future.

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