Infinite and Infinitesimal in Mathematics, Computing and Natural Sciences

Grand Hotel San Michele – Cetraro (CS), Italy
17-21 May 2010

BOOK OF ABSTRACTS

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International Workshop

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Infinite and Infinitesimal in Mathematics, Computing and Natural Sciences

17–21 May 2010 — Grand Hotel San Michele, Cetraro (CS), ITALY

Dear Participants,

Welcome to the International Workshop *Infinity–2010* 'Infinite and Infinitesimal in Mathematics, Computing and Natural Sciences'. The goal of the Workshop is to create a multidisciplinary round table for an open discussion on modeling nature by using traditional and emerging computational paradigms. Mathematics and natural sciences offer discrete and continuous models to describe space, processes, and events occurring in nature. Very often both approaches use notions of infinite and infinitesimal in order to create coherent models. It is assumed that it is possible to work with infinitesimal quantities and/or to execute an infinite number of steps in algorithms. However, our abilities in computing are limited and only a finite number of computational steps can be executed.

The Workshop discusses all aspects of the usage of infinity and infinitesimals in mathematics, computing, philosophy, and natural sciences. Fundamental ideas from theoretical computer science, logic, set theory, and philosophy meet requirements and new fresh applications from physics, chemistry, biology, medicine, and economy. Researchers from both theoretical and applied sciences participate in the Workshop in order to use this excellent possibility to exchange ideas with leading scientists from different research fields.

A special attention is dedicated to the new methodology allowing one to execute numerical computations with finite, infinite, and infinitesimal numbers on a new type of a computational device—the Infinity Computer (EU patent 1728149). The new approach is based on the principle 'The part is less than the whole' introduced by Ancient Greeks that is applied to all numbers (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite). The new methodology evolves Cantor's ideas in a more applied way and introduces new infinite numbers that possess both cardinal and ordinal properties as usual finite numbers. It gives the possibility to execute numerical computations of a new type and simplifies fields of mathematics where the usage of the infinity and/or infinitesimals is necessary.

The Organizing Committee thanks sponsors of the event for their support: University of Calabria (Italy); Department of Electronics, Computer and System Sciences (Italy); International Association "The Friends of the University of Calabria" (Italy); Institute of High Performance Computing and Networking of the National Research Council (Italy); Italian National Group for Scientific Computation of the National Institute for Advanced Mathematics "F. Severi", and Italian National Bank of Labor of the BNP Paribas Group.

We wish to all participants a successful work and hope that the Workshop will give you a lot of inspiration leading to new important results in your scientific fields.

The Organizing Committee

Table of Contents

Tutorials and Invited Lectures

Actually Infinite Prices in Stochastic Pure Exchange Model with Perfect Foresight

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Keywords. Mathematical economy; rational expectations; infinite prices.

Stochastic pure exchange general equilibrium model is considered. The economy consists of finite number of consumers. Each consumer at each period of time receives random endowment of a single commodity.

Each consumer plans consumption to maximize expected utility subject to intertemporal budget constraints. There are as many budget constraints as the number of realizations of stochastic process. Equilibrium conditions consist of material balances at each moment of time.

We've proved that for each set of the terminal prices there exists appropriate equilibrium [1]. Also equilibria with the infinite price exist.

Two techniques could be used to describe equilibria with the infinite prices. The first one is a hierarchical prices [2]. The second technique is infinitesimals and a non-standard analysis [3]. We've proved the equivalence of the equilibria formulated in terms of non-standard analysis and in terms of hierarchical prices.

The most intriguing are the equilibria with infinite prices. At equilibrium trajectory price cannot fall to zero but can jump to infinity with positive probability. When price jumps to infinity all debts and savings depreciate and "default" occurs. Equilibria with the infinite prices frequently Pareto dominate "finite" equilibria.

Acknowledgements.

This research was supported by Russian Foundation for Basic Research, project num. 09-01-13534-ofi-c.

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Operations Research and Grossone

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Keywords. Mathematical programming; simplex method, anticycling method; data envelopment analysis; nonlinear programming.

In this talk we will discuss some possible applications of ① in various fields of Operations Research and Mathematical Programming. The aim is to improve the efficiency of traditional methods widely used in linear and nonlinear programming.

First of all we will show how the use of ① can be beneficial in eliminating the cycling phenomena in the simplex method for linear programming. Convergence of the simplex method requires that specific techniques for avoiding cycling are applied, such as perturbation of the right-end-side term or Bland's rule. These techniques are all fairly complicated to use in practice, especially when the matrix form of the simplex method is used. We propose here an anticycling method that utilizes the new calculation system by perturbing the right-end-side term with a infinitesimal quantity expressed by positive powers of ①.

Another application of ① that will be presented is related to the Data Envelopment Analysis (DEA) methodology, for evaluating the efficiency of Decision Making Units (DMU). In the basic version proposed by Charnes, Cooper Rhodes in 1978 the use of a infinitesimal non-archimedean quantity ϵ is required. We will show how the use of negative power of ① allows to achieve the same theoretical results and thus the efficiency of a single DMU can be easily obtained by solving a single linear programming problems using the new arithmetic based on ①.

The third and final application we will present is in the context of the nonlinear programming. Exact continuously differentiable penalty methods have been studied extensively in the past. Here we propose a novel penalty function that depends on ①.

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Infinity in Art and Mathematics

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Keywords. Art; infinity; mathematics; Escher.

The Dutch Graphic artist Maurits Cornelis Escher for most of his life was attracted by the problem of symmetry, of the periodic tiling of the plane, of tessellations. He was fascinated by Moorish works at the Alhambra of Granada in Spain in the 1920s and then he rediscovered them after he left Italy in 1936.

Escher used the periodic tiling of the plane because he had a dream, to make possible Hamlet's vision: 'I can be bounded in a nut shell and become the king of infinite space.' Escher described in great details how he reached the problem of infinity and how he tried to solve it. 'For a long time I have been interested in patterns with motives getting smaller and smaller till they reach the limit of infinite smallness. The question is relatively simple if the limits is a point in the enter of the pattern. I was never able to make a pattern in which each blot is getting smaller gradually from a center towards the outside circle limit.'

The encounter between Coxeter and Escher was a very significant for both. Escher saw a model of Poincare's hyperbolic geometry in a book of Coxeter and he wrote to him for 'simple explanations'. From the meeting with Coxeter the series of engravings 'Circle Limit', described by Escher in his first book as 'Infinity of Number'. I would like also to describe the animation technique which adding time and movement to the images of Escher try to contribute to Escher's dream of reaching infinity.

Acknowledgements.

This research was supported by the PRIN 2007 grant 'Matematica, arte, cultura: Possibili connessioni'.

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Continued Fractions as Dynamical Systems

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Keywords. Continued fractions; dynamical systems; number representation.

One of most promising attempts to improve the arithmetic of the computers is to use the string arising from the continued fraction expansion. In this representation in fact a number is represented by a string very much similar to the usual decimal representation: $N = a_0 a_1 \dots a_n$. Here, however the string does not represent the number $N = a_0 10^n + a_1 10^{n-1} \dots a_n 10^0$ but a more involved quantity. The advantages are many:

- rational numbers are represented by finite strings;
- quadratic irrational numbers are represented by periodic strings, just as the rationals in the decimal representation;
- the continued fraction expansion is *base free*, that is, the value of the coefficients a_i is invariant with respect to a change of base;
- transcendental numbers are easily and well approximated.

It is plain that if an arithmetic based on such string, even with a finite representation, could be implemented on computers, many of the well known limitations would be overcame. The problem is that the arithmetic with such numbers is not easy. At the moment, there are many attempts, which are, according to the authors, very promising. Here we will not discuss about this aspect, rather we shall deal with the behaviour of the dynamics associated with the continuous fraction expansion. Even if this is a problem studied in depth during the last three centuries, there is still room for new results especially towards enlightening the trade off between the continuous and the discrete aspect of the real line. Our approach will use the dynamical system theory. The attractivity of the set of reduced quadratic polynomials, denoted by S_2, permits to easily derive the fundamental Lagrange theorem. By establishing the relations between the matrices describing the dynamical system in two and three dimensions, the Pell equations arise in a very natural way. Moreover, the study of the length of the orbits in S_2 requires other classical results in number theory.

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Percolation and Infinity Computations

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Keywords. Site percolation; infinite clusters; gradient percolation.

We consider a number of traditional models related to the percolation theory [1] using the new computational methodology introduced in [2, 3]. It has been shown that the new computational tools allow one to create new, more precise models of percolation and to study the existing models more in detail. The introduction in these models new, computationally manageable notions of the infinity and infinitesimals gives a possibility to pass from the traditional qualitative analysis of the situations related to these values to the quantitative one. Naturally, such a transition is very important from both theoretical and practical viewpoints.

The point of view on Calculus presented in this paper uses strongly two methodological ideas borrowed from Physics: relativity and interrelations holding between the object of an observation and the tool used for this observation.

Site percolation and gradient percolation have been studied by applying the new computational tools. It has been established that in infinite system phase transition point is not really a point as with respect of traditional approach. In light of new arithmetic it appears as a critical interval, rather than a critical point. Depending on "microscope" we use this interval could be regarded as finite, infinite and infinitesimal short interval.

Acknowledgements.

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Infinitesimals and Infinities in the History of Mathematics

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Keywords. Infinitesimals; indivisibles; infinite numbers.

Infinitesimal lines make an appearance in Archimede's work, in the 3^{rd} century b. C., in the calculus of areas such as that of the parabolic segment. A line according to Euclid's definition should have been a length without largeness, or thickness, but in the applications their largeness had to be positive, though negligible, in order that "all the lines" could fill a figure. The locution "all the lines" is due to Bonaventura Cavalieri (1598–1647); in the sixteenth century he brought to light Archimedes' method and used it as a heuristic principle: two plane figures have the same ratio as "all their lines" with respect to an arbitrary direction. In latin: *ut unum ad unum, sic omnia ad omnia*.

Galileo Galilei (1564–1642) was opposed to indivisibles for philosophical reasons; he opposed the metaphysical atomism according to which the continuum is a set of points. Since he was convinced that there was only one type of infinite, points could not add up to segments of different length, and lines could not be used as a measure of figures of different areas. He was compelled however to use them in a few proofs relative to uniform and uniformly accelerated motions; there wasn't yet a language for motion and velocity, but the geometrical one, and the cumbersome theory of proportions.

Leonhard Euler (1707–1783) was the first to use infinite numbers in arithmetical and analytical researches. He represented the number line as $1, 2, 3, \ldots, \infty$, with $\omega = \frac{1}{\infty}$ as infinitesimal. He considered infinite series as polynomials of degree ∞, and applied to them the associative and commutative properties.

His intuition never erred, but the laws for these new numbers were difficult to recognize; Gottfried W. Leibniz (1646–1716) thought they could be the same as for finite numbers, but this clearly was impossible without further specifications. Eventually they were abandoned, in favour of the definition of limit.

After the age of rigour in analysis, there was no place for enlargements of the number system, but with non archimedean fields, as Giuseppe Veronese (1854–1917) tried to foster. But Georg Cantor (1845–1918), who created the theory of the infinite, was absolutely contrary to infinitesimals.

They had to wait till the second half of the twentieth century to be salvaged by a clever use of logical languages and machinery by Abraham Robinson (1918–1974). They can be made to satisfy the same laws as the real numbers, so long as these laws are expressed in the first order language of the theory.

The above is jus a sketch, we will go deeper into each of the historical episodes in our exposition.

An Application of Grossone to the Study of a Family of Tilings of the Hyperbolic Plane

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Keywords. Grossone; tilings; hyperbolic geometry.

The paper deals with an application of ① to an important family of tilings of the hyperbolic plane. Namely, we consider the tessellations which are defined from a regular polygon by replication the figure by reflections on its sides and then, recursively, by replication of the images in their sides.

We remind the way described in [1, 2] with which it is possible to enumerate the tiles of a part of the hyperbolic plane, especially in two of these tilings, tilings $\{5, 4\}$ and $\{7, 3\}$ called here the pentagrid and the heptagird respectively. Each enumeration gives ① tiles and we describe two possible decompositions of the tiling into such parts giving rise to different values on the number of observable tiles. Here, we use the term observable in the sense given by Yaroslav Sergeyev in his seminal works, see [3] for instance. Also, we consider the observable area as the area of the tiles is easy to be computed. This also gives us a precise estimate of the observable thanks to the use of ①. It is known that the pentagrid and the heptagrid are indeed connected by a deep arithmetic property, see [2]. Now, one decomposition gives us the same observable area for both tilings. The numbers of observable tiles are different, the area of the basic polygons are different but the total observable area is the same. This gives another way to express the connection between these tilings which was obtained by purely arithmetic considerations.

The counting results and the above remarks can be extended to two infinite families of tilings of the hyperbolic plane. We also extend the counting results as much as possible to all tessellations of the hyperbolic plane from a regular polygon.

Acknowledgements.

I am in debt to Yaroslav Sergeyev for his interest to this work.

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Numerical Computations with Infinite and Infinitesimal Numbers: Methodology and Applications

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Keywords. Infinite and infinitesimal numbers; infinite sets; Infinity Computer.

This tutorial introduces a new methodology allowing one to execute numerical computations with finite, infinite, and infinitesimal numbers (see [1, 2]) on a new type of a computer—the Infinity Computer (see the European Patent [3]). The new approach is based on the principle 'The part is less than the whole' introduced by Ancient Greeks. It is applied to all numbers (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite). It is shown that it becomes possible to write down finite, infinite, and infinitesimal numbers by a finite number of symbols as particular cases of a unique framework different from that of the non-standard analysis.

The new methodology (see survey [3]) evolves Cantor's ideas in a more applied way and introduces new infinite numbers that possess both cardinal and ordinal properties as usual finite numbers. It gives the possibility to execute computations of a new type and simplifies fields of mathematics where the usage of the infinity and/or infinitesimals is necessary (e.g., divergent series, limits, derivatives, integrals, measure theory, probability theory, fractals, etc.). Numerous examples and applications are given. The First Hilbert Problem is studied in depth (see [4]). Numerous examples are given.

The first software application using the Infinity Computer technology—Infinity Calculator—is presented during the talk.

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Relativity in Mathematical Descriptions of Automatic Computations

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Keywords. Theory of automatic computations; observability of Turing machines; infinite sets.

The tutorial deals with the problem of the mathematical descriptions of automatic computations by taking the Turing machine as the reference computational model [4]. In particular, the problem is approached using a new methodology [2] which emphasizes the role of the philosophical triad – the researcher, the object of investigation, and tools used to observe the object, and makes it possible to investigate the interrelations that arise between automatic computations themselves and their mathematical descriptions when a human (the researcher) starts to describe a Turing machine (the object of the study) by different mathematical languages (the instruments of investigation). Along with traditional mathematical languages using such concepts as 'enumerable sets' and 'continuum' to describe the potential of automatic computations [1, 4], a language introduced recently and the corresponding computational methodology which allows to measure the number of elements of different infinite sets is exploited [2]. The new mathematical language allowed the authors to obtain some results regarding both the sequential computations executed by the Turing machine and the produced computable sequences [3]. In the tutorial, deterministic and non-deterministic machines are also described using both the traditional and the new languages and the obtained results discussed and compared.

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A New Approach to Classical and Quantum Mechanics

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Keywords. Classical mechanics; probability distribution; quantum mechanics.

We argue that the Newton–Laplace determinism in classical world does not hold and in classical mechanics there is fundamental and irreducible randomness. The classical Newtonian trajectory does not have a direct physical meaning since arbitrary real numbers are not observable. There are classical uncertainty relations, i.e. the uncertainty (errors of observation) in the determination of coordinate and momentum is always positive (non zero).

A "functional" formulation of classical mechanics is suggested. The fundamental equation of the microscopic dynamics in the functional approach is not the Newton equation but the Liouville equation for the distribution function of the single particle. Solutions of the Liouville equation have the property of delocalization which accounts for the time irreversibility. The Newton equation in this approach appears as an approximate equation describing the dynamics of the average values of the position and momenta for not too long time intervals. Corrections to the Newton trajectories are computed. This approach leads to a new computational scheme for various problems in mechanics.

The usual Copenhagen interpretation of quantum mechanics assumes the existence of the classical deterministic Newtonian world. Here an interpretation of quantum mechanics is attempted in which both classical and quantum mechanics contain fundamental randomness. Instead of an ensemble of events one introduces an ensemble of observers.

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Summation of Infinite Series

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Keywords. Infinite series; divergent series; convergence of infinite series.

We discuss several topics related to the summation of infinite series in the new infinity paradigm. We show that the new paradigm can be extremely useful for simplifying technical issues related to computing the values of convergent series and to the summation of divergent series; see [1, 2] for the main concepts and methods related to the divergent series.

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Theory of Probability in the New Infinity Paradigm

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Keywords. Probability theory; probability distribution; limit theorems; entropy.

We discuss how the theory of probability is looking like in the new infinity paradigm. The main idea is that there are no arbitrary measures any more; all measures are discrete with either finite or infinite number of support points. This makes the probability theory simpler and more transparent. Let us make a few examples:

- to define a measure we no longer need to define a σ-algebra as the set of all subsets of X would suffice, where X is the set of support points;
- there are no paradoxes of the type $\mathbb{P}(\xi = x) = 0$ for all x;
- the treatment of conditional expectations becomes much simpler;
- there is no inconsistency between the discrete entropy $-\sum_i p_i \log p_i$ and its continuous counterpart $-\int p(x) \log p(x) dx$.

The crucial point, however, is how do the limit theorems of probability theory change. There is not much change in what relates to the laws of large numbers. There is, however, a big change related to the formulation and interpretation of the central limit theorem, the law of iterated logarithm and many other theorems.

Consider, as an example, the Moivre-Laplace theorem which states, roughly speaking, that the binomial distribution of the number of "successes" in n independent Bernoulli trials with probability p of success on each trial is approximately a normal distribution with mean np and standard deviation $\sqrt{np\,(1-p)}$, if n is very large. In the new paradigm, however, there is no normal distribution which is supported on the real line $\mathbb{R} = (-\infty, \infty)$. Instead, there are many 'discrete normal distributions'; in the language of traditional mathematics, these distributions can be considered as approximations to the 'continuous normal distribution'. All these 'discrete normal distributions' can be distinguished firstly, by their supports and secondly, by their weights.

In the case of the Moivre–Laplace theorem, the support of the discrete normal distribution is the set of all nonnegative integer numbers rather than $(-\infty, \infty)$ in classical statistics. This distribution is simply Binomial with parameters ① and p; it has mean ①p and standard deviation $\sqrt{①p\,(1-p)}$. One of the important practical issues is that this formulation immediately resolves the paradox of classical statistics according to which there is always a positive probability that the number of successes in n independent Bernoulli trials can be negative. This paradox is sometimes a serious inconvenience for practitioners.

Regular Presentations

Method of Formalizing Computer Operations for Solving Nonlinear Differential Equations

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Keywords. Computer arithmetical operations; rank-transfer procedure; basic finite difference scheme; nonlinear differential equations.

A computer model formalizing the fundamentals of operating with numbers is proposed. The main aspects are as follows: keeping a finite number of ranks and the rank-transfer procedure. A new method based on this model is suggested in which the solution of a differential equation is represented in an explicit form as a segment of the power-series in powers of the step of the argument. The first term of this series is able to hold the infinitely large number due to vanishing the step of the argument. The next term is the linearization term which is generally of the order of unity. The other terms hold infinitely small values. The algorithm generates the scheme which approximates the basic finite difference scheme which in turn approximates the equation under consideration. The formalization of the computer operations can construct interesting stochastic and fractal objects related to the senior ranks (see [1]). Thus the use of the probabilistic methods allows us to exclude intermediate levels of our scheme. Such an approach allows us to construct explicit solutions for the problems which can be not solved in quadratures. We demonstrate application of the method for solving the Riccati equation that leads to solution in form of the continued fractions. Also we apply the method at hand for solving several systems of the kinetic equations and for solving the van der Pol oscillator problem which can be exposed in a pair of differential equations.

Acknowledgements.

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Potential Infinity... Not Accessible

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Keywords. Computable vs measurable numbers; structural complexity.

In this paper we explore, beyond our work in [1, 3], an unexpected limitation of physical theories whenever scientists are no more powerful than Turing machines: The Turing experimenter, in the world of continuous physical variables, *(even) theoretically equipped with infinite precision instruments*, does not have access to the values of quantities above a finite number of bits, rendering the infinite inaccessible *even) in gedankenexperimente.*

Having investigating physical systems from all branches of Physics, we reached the conclusion that *all physical experiments of a specified class can be seen as oracles (see [2]) to a Turing machine exhibiting a computational power (in polynomial time) of* $P/log*$. Moreover, we have the following limiting result: *There are uncountably many values of each physical quantity* ζ *so that, for any scientist with a specified computable schedule, having access to the required equipment to measure* ζ*, there is an* n *so that scientist will never know the first* n *binary places of* ζ. We have reasons to believe that these results constitute a full characterization of an analog system, no matter its physical substrate.

Finally, we formulate in this paper a law of information processing in Nature: *no scientist can retrieve information from a physical system in less time (lower bound) than exponential on the size of the precision in bits; moreover, the upper bound is infinite.*

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Computing with Nonlinear Dynamics: A CNN Approach

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Keywords. Wave computing; nonlinear dynamics; Cellular Neural Networks (CNN), Embodied PDE into CNN.

Nonlinear Partial Differential Equations (PDEs) represent a powerful mathematical tool for describing a wide class of phenomena, ranged from the waves in shallow water to the dynamics of plasmas, the toda lattice, etc. The key feature of many of these equations is the presence of soliton-like solutions. It is useful to mention that the solitons, observed for the first time in 1834 by the Scottish naval engineer John Scott Russell and identified as such by Zabuski and Kruskal in 1965, have the characteristic to emerge from a balance between nonlinearity and dispersion and to show simultaneously both the wave and particle behaviour. It was shown that PDEs behavior can be well modeled by CNNs, Cellular Neural Networks, introduced by Chua and Yang in 1988. CNNs are continuous-time and discrete-space models, consisting of a set of ODEs, which have the important property of being easily implementable in VLSI. A new computational paradigm, known as wave computing, has originated. Cellular neural network chips achieve extraordinary performances (in the order of Tera operations/sec), due to the massively parallel analog computation performed by CNN-based architectures. Recent studies have also shown the possibilities of implementing PDEs on the CNN-based models. Compared to traditional studies, this is a very innovative approach, because it allows both to analyze many nonlinear phenomena in controlled laboratory settings and, eventually, to use them at the engineering level. In this paper we present CNN transmission lines based on nonlinear components, that simulate the behavior of various PDEs, with a particular reference to the Korteweg-de Vries (KdV) equation and its generalized forms. Several different experiments are presented and the behavior of emerging solitons is particularly analyzed. The most innovative aspect is connected to the intersection of several transmission lines. In this case, the dynamics of solitons shows significant differences from the traditional dynamics, representing an array of very different and complex behaviors. Different circuit topologies and some new nonlinear phenomena are introduced. This research can open new foundations for multidimensional interaction between solitons as well as inspire to new forms of unconventional computing, easily implementable on VLSI. Moreover, this work describes new "brain-like" approaches to computation, that use the wide variety of nonlinear phenomena.

On Implementation Aspects of the Infinity Computer

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Keywords. Infinity Computer; software and hardware implementation.

Computational abilities of modern computers determine superiority of one product with respect to another. Traditional computers are able to deal with only finite numbers because arithmetics developed for infinite numbers leave undetermined many operations or represent these numbers by infinite sequences of finite numbers (e.g., nonstandard analysis approaches). Thus, in spite of the key role of infinitesimals and infinite in physics and mathematics (e.g., derivatives, integrals, and differential equations, see [1, 2]), the fields of natural sciences related to infinite and infinitesimals remain purely theoretical.

The unconventional computational paradigm introduced in [3, 4] can be used for creating a new type of computer—Infinity Computer—able to execute numerically operations with infinite, finite, and infinitesimal numbers. The key idea is the usage of a new positional system with infinite radix allowing one to express finite, infinite, and infinitesimal numbers by a finite number of symbols in a unique framework (see [4]). In the talk, possible ways for developing the Infinity Computer simulator as well as its hardware prototype are discussed.

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On Beppo Levi's Approximation Principle

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Keywords. Infinite choices; axiom of choice; approximation principle.

It is well–known that, between the end of the 19th and the beginning of the 20th century, a group of Italian mathematicians played a primary role in a rising controversy about allowing infinite choices in mathematics. In particular, criticisms against references to implicit principles of that sort were made by G. Peano, R. Bettazzi, and B. Levi. This fact is acknowledged by some of the most authoritative books on the choice issue such as G. H. Moore [2] and H. Rubin and J. E. Rubin [3].

It is less known, if not basically unknown, that Levi himself, much later, made a specific proposal for substituting AC by what he called the Approximation Principle (AP). Levi made very little effort for publicising this proposal, as he wrote all of his papers dealing with it in an old–fashioned Italian. It is not surprising, thus, that besides some contemporary of Levi's (some who actually worked on AP like T. Viola and G. Scorza-Dragoni, others who only commented it such as U. Cassina and A. Faedo), and some notable exceptions among scholars (G. Lolli's editorial note in [1, vol. I, pp. LXVII–LXXVI], and Moore's [2, pp. 244]), basically nothing is known about this part of Levi's work.

The present contribution aims at reviving a discussion on AP, both in a philosophical and in a metamathematical direction. In particular: (i) we analyze the philosophical surroundings of the principle, namely Levi's (sort of a) conceptualist view of mathematics as based on assuming as primitive a domain of objects each of which one can pick an element from; (ii) we use the latter in order to provide a practicable formulation of AP (the first serious attempt in this sense, Moore's one we have referred to above appears to be affected by several misunderstandings); (iii) as a case study, we offer an application of the principle in the proof of a fundamental property of metric spaces. Finally, some further comment about the foundational status of AP, as well as about future work, will be given.

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Model Checking Time Stream Petri Nets: An Approach and an Application to Project Management

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Keywords. Time stream Petri nets; timed automata; Uppaal; model checking; project management.

Merlin and Farber Time Petri nets (TPNs) have been proven to be a very convenient formal tool for expressing timing constraints in time-dependent systems. TPNs associate transitions with time pairs. Property analysis rests on a time reachability graph [1] which provides a (possibly) finite representation of the (normally) infinite dynamic behaviour of a TPN. To deal with infinite firing times of transitions, graph nodes are state classes holding a net marking and a firing domain (clock inequalities system or time zone) reflecting timing constraints, and edges are labelled with firing transitions.

In this work Time Stream Petri Nets (TSPNs) [2] are considered, which are more powerful than TPNs. They permit detection of constraint violations both at the task (firing sequence) level and at the single transition level. Timing constraints have the form of time pairs and are associated with arcs only. A weak synchronization model applies to arcs but, as in TPNs, a strong synchronization model is associated with transition firings. Transitions can be annotated with one of several firing rules which give great flexibility to the modeller.

The paper outlines an approach to TSPN modelling and verification (M&V) which is based on a preliminary transformation of a source model into the terms of timed automata of popular Uppaal [3] tool, which is then model checked. An application to project management, that is M&V of CPM/PERT like techniques [4] is then provided. Novel in this application is the possibility for activities to have uncertainty dense time intervals as durations.

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The Analyst Revisited
(From Berkeley to Brouwer)

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Keywords. Infinitesimal calculus; intuitionistic logic; distinguishability.

The aim of this paper is to show that there exists a link across centuries between two historic quarrels in the foundation of mathematics. The first dispute involved Bishop Berkeley [1] of Cloyne and the inventors of the calculus. The second dispute involved L.E.J. Brouwer [2] and the debate over David Hilbert's second problem concerning the completeness and consistency of the axioms of mathematics.

The two quarrels will be detailed, it will then be explained how they are intimately connected giving a specific original proof, and then the connection in the general case will be explored. It will be discussed how significant this is for mathematics and the philosophy of mathematics, and the paper concludes with a sampling from the metaphysics of the thinker René Guénon [3] and also with a pointer to future research.

The central logical link that is constructed should be a useful addition to the philosophy of mathematics and, as a consequence, to mathematics itself. The paper draws on the work of C.S. Peirce, J.L. Bell [4] and R. Guénon to provide a framework for and alternative views on the debates at hand.

This interdisciplinary paper touches on quite a number of the workshop topics, these being: the foundations of mathematics, logic and infinity, the philosophy of mathematics, and infinitesimals.

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A Metaphysical Interpretation of Einstein's Relativity Theory

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Keywords. Einstein's tensor equations; cosmological constant; infinity in potency and in act; Gödel's universes; vacuum and immaterial solutions.

This paper deals with a possible metaphysical interpretation of Einstein's Relativity Theory. More precisely, the main aim of this work is to give a novel explanation concerning the existence of what physicists call dark energy and cosmologists quintessence. Although philosophical considerations can be deduced from the investigation of this elusive energy, the approach presented here is essentially scientific, being based on Einstein's tensor equations and on the corresponding metrics from a metaphysical and meta-mathematical point of view.

The role played by the cosmological constant $\mathbf{\Lambda}$ is of course fundamental. In particular, it is shown that $\mathbf{\Lambda}$ must be necessarily considered as a space-time function. Since the latter assumption violates the local conservation of energy, the classical vacuum solutions of Einstein's equations must be generalized in a new concept named 'immaterial solutions', thereby defining the corresponding energy-fields as 'immaterial substances'. By introducing a meta-mathematical rule for the computation of undetermined forms, one can overcome both the paradoxes derived by the existence of closed time-like curves in Gödel universe and the quantistic dilemma concerning the range of possible values for $\mathbf{\Lambda}$. Moreover, the classical cosmological alternatives $\mathbf{\Lambda} = \mathbf{0}$ or $\mathbf{\Lambda} \neq \mathbf{0}$ and hence, the creation, the expansion and the asymptotical behaviour of our universe can be seen in terms of two distinct hypotheses concerning the 'destiny' of biological life.

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Possible Applications of the Infinity Theory in Heat Processes with Impulse Data

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Keywords. Infinity theory; impulsive initial boundary value problem; heat equation; eigenvalue; eigenfunction; generalized solution.

Two boundary value problems for heat conduction equations are considered. Initial conditions of these problems are Dirac delta functions. The heat equation of one problem has piecewise-constant coefficients. The process of the distribution of the temperature in two-layered composite is described by this equation. The Fourier series expansion method is applied in order to solve these problems. The formal (generalized) solutions are constructed in the form of the Fourier series. The possibility to apply the infinity theory to explain the behavior of the obtained solutions are discussed.

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Zeno Paradox and Quantum Mechanics

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Keywords. Continuum; discrete; Heisenberg indetermination principle.

We review the ideas that lie at the foundations of the famous Zeno Paradox on motion, known as the Paradox of 'Achilles and the Turtle', according to which Achilles would never be able to reach the Turtle since this would require an infinite number of steps in Space and Time. This famous Paradox has been the basis of long debates about the nature of motion, its possibility in Euclidean space and the very nature of infinitesimals and infinite processes.

It is well known that this paradox — together with other fundamental issues related also to Democritus Atomism and the very nature of Space and Time — was eventually solved by a mixture of the following ideas: 1) Space is a continuum, ruled by Real Numbers rather than by Rational Numbers (as it was supposed until the discovery of Irrationals); 2) Euclidean Geometry is based on Continuity; 3) Motion can in fact be decomposed into a non-numerable sequence of instantaneous positions of equilibrium (D'Alembert's Principle and Lagrange's Analytical Mechanics); 4) it is possible to sum-up numerable sequences of infinitesimals to obtain convergent series. According to the Euclidean vision and to the belief that Nature has to be ruled by Real Numbers, after the time of Greeks our way of doing Mathematics has progressively abandoned the idea that the Physical World might have a discrete nature and the method has privileged the paradigm of continuity.

The paradigm of 'continuity' has generated a number of issues that, one side, have given the birth to modern Mathematics and, on the other, to deep speculations about the very notions of infinitesimals and infinities, still nowadays object of debate. Along with 'standard analysis' also 'non-standard analysis' has been developed and new fresh ideas due to Ya. Sergeyev have introduced new ways of calculations with infinities and infinitesimals.

Nevertheless, the birth of Quantum Mechanics has also led — through the well known 'Heisenberg's Indetermination Principle' — to challenge again the idea that nature is continuous rather than discrete. Quantum Mechanics seems to point out that no real measure can be obtained beyond a limit determined by Planck's Constant. Nature is formed by 'quanta'. According to this new vision Zeno Paradox can be revisited by saying that Achilles would never reach the Turtle since he will eventually surpass it without being able to see it again in positive time, so that seeing it requires an inversion of the time arrow.

Complex Integrated Systems Modelling with Both Continuous and Discrete Timescales

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Keywords. Unified systems modelling; discrete & continuous time; mathematical foundations of systems; systems architecture framework.

We study complex integrated systems, i. e. complex systems whose architecture can be defined by a succession of integration and abstraction processes. Our general purpose is to provide a formalized and generic architecture framework for the design of complex systems. In the scope of this workshop, we focus on the most fundamental part of our work, centered on the definition of semantics of such systems, consistent with their architecture. Our model gives a broad framework for all timescales, in particular those based on infinitesimal numbers.

After defining a system through formal intuitive conditions, we show that it can be modelled in a generic framework, mixing both discrete and continuous timescales (that can be changed by an operator called "abstraction"). A system is modelled as a 5-uplet $S = \big(T,\ \mathbb{Q},\ \mathbb{Q}_0,\ \mathcal{F},\ \mathcal{Q}\big)$, which can be interpreted, within a model of time T, as the interconnection of two functions (generalizing the discrete mechanism of Turing machine): an internal behavior $\mathcal{Q}$ (describing the evolution of the state of a system) and a function $\mathcal{F}$ (modelling the relation between the inputs and the outputs, according to the state), allowing to handle in a single abstract object multiple instances belonging to a consistent family.

This work intends to provide a generic framework for systems modelling, encompassing in particular a model of systems using non-standard analysis to mix continuous and discrete time previously introduced in [1]. We also provide operators allowing to define architecture of such systems; in particular, the *abstraction* allows to consider a higher-level model of a system, making it possible to shift from continuous to discrete time.

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Julius Koenig Sets as Higher Infinity

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Keywords. Infinity; Koenig sets, nonstandard universes.

At the third international congress of mathematicians in Heidelberg, a Hungarian mathematician J. König shocked mathematical community by proving that the continuum could not be well-ordered. In fact König's own argument, presented in [4], was correct, but unfortunately König noncritically assumed an earlier wrong claim by F. Bernstein.

Further arguments on the base of Bernstein's wrong claim led König to a conclusion even more striking: the cardinality of the continuum exceeds any "aleph" in Cantor's sequence of cardinals of wellorderable sets, [4, p. 180]. Since then, any hypothetical non-wellorderable set X the cardinality of which exceeds the cardinal of each wellorderable set, is called *König set*.

The existence of a König set, in its first part (non-wellorderability) contradicts the axiom of choice, and in its second part (being above every aleph) contradicts basic set theoretic axioms of **ZF** even in the absence of the axiom of choice. Therefore it was considered as a problem to find a suitable environment for König sets. By necessity such an environment cannot fully satisfy **ZF**. Recent results of Kanovei and Shelah [3], generalized in [1] and [2, Ch. 4] demonstrated that completely saturated nonstandard extensions of the ZFC universe allow to adequately model König sets.

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Applications of Non-Standard Methods in Fourier Analysis

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Keywords. Non-standard analysis; Fourier series; Fourier transform.

A rigorous treatment of infinitesimal small and infinitely large numbers and the definition of hyperreal numbers $^*\mathbf{R}$ was first given by Abraham Robinson in 1966. The hyperreals were originally defined with model theory but can also be obtained by an ultrapower construction with equivalence classes of sequences of real numbers. There are also axiomatic approaches for a non-standard set theory extending the standard Zermelo–Fraenkel (ZFC) theory.

We first provide a short introduction to non-standard analysis. The non-standard real numbers $^*\mathbf{R}$ (and other non-standard sets) are constructed via the ultrapower method and examples of infinitesimal and infinite non-standard numbers are given. The axiomatic approach using Internal Set Theory (IST) is also mentioned. The transfer principle shows that many statements can be extended from $\mathbf{R}$ to $^*\mathbf{R}$.

Then, some basic constructions from calculus are presented in a non-standard formulation: convergence, continuity, the derivative and the integral.

The main part of this contribution discusses applications to Fourier analysis. A unified description of the Fourier coefficients, the Fourier Transform and the Discrete Fourier Transform is given in the framework of non-standard analysis. Infinitesimal and infinite numbers turn out to be advantageous and some convergence arguments are easier in the non-standard context. It is shown how non-standard methods can be employed to examine the pointwise convergence of the Fourier series and the inversion formula.

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A Hybrid CA Approach for Natural Sciences Simulation

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Keywords. Microscopic and macroscopic systems simulation; cellular automata; quantum computing.

Cellular Automata (CA), one of the most challenging computational paradigms in microscopic and macroscopic [1] complex systems simulation were successful addressed also by using a modified CA classical approach [2].

The presentation will discuss related aspects in applying the CAN 2 [2] approach for the simulation of natural sciences such as: debris flows, superconductive devices [3] and forest fire simulation [4].

Advantages and limitations are introduced when both microscopic and macroscopic dynamics are taken into account justifying the introduction of hybrid components between singular cellular automata, i. e. a network in which global behavior and local interactions can coexist with side effects in computational parallelism addressing.

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Philosophical Aspects of a New Approach to Infinity: From Physics to Mathematics

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Keywords. Philosophy of mathematics; grossone approach to infinity; infinite and infinitesimal numbers; infinite sets.

Problems related to infinity take a particular place in science. It is sufficient to mention the First Hilbert problem. The point of view on infinity accepted nowadays takes its origins from the ideas of Georg Cantor who has shown that there exist infinite sets having different cardinalities. However, it is well known that Cantor's approach leads to some counterintuitive situations.

Recently a new methodology using an infinite unit of measure — grossone — has been introduced in [1, 2]. It gives a possibility to work with finite, infinite, and infinitesimal quantities numerically on a new kind of a computer introduced in [3]. This method allows one to solve some problems related to infinity (see [4]).

The goal of this study is to analyze philosophical aspects of this efficient approach and to discuss its methodological foundations and their origins.

The new approach uses strongly two ideas borrowed from Physics: relativity and interrelations holding between the object of an observation and the tool used for this observation. These aspects of the method are discussed in detail.

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Metaphysics of Infinity: The Problem of Motion

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Keywords. Infinite convergent series; complex infinity; founding principle; Anaxagorian theory of motion.

Since the time of the Greek philosopher Zeno who formulated his paradoxes of motion, our understanding has failed to comprehend motion throughout the ages. The laws of motion are defined, relativity and quantum theories are discovered, our telescopes collect light from the deepest recesses of the physical universe and our spacecrafts travel beyond the limits of our solar system — but the nature of motion from a to b by passing through an infinite convergent series of sub-distances in a finite time remains unintelligible!

Throughout our lecture we will show how the complex idea of infinity dissolves the motion paradox and determines its founding principle. At first we will investigate Aristotle's different analytic solutions proposed in his *Physics VI* and show why analytic principles of thought can in no way help us in comprehending continuous motion from a to b. Because motion resides outside the analytic principles of our understanding, we return to the dawn of science and philosophy in order to find fresh insights and inspiration. Based on Anaxagora's theory of motion (Greek Ionian philosopher of $6th$-$5th$ century BC) we propose a synthetic solution to the problem of motion that enable us to move from a to b, from the unlimited convergent series to its limit. We take in turn this synthetic solution as the founding principle of continuous motion.

Finally once the founding principle of motion is identified, we will investigate the sweeping consequences that this first principle has over the Aristotelian theory of the physical body according to which no finite body has infinite power such as for instance executing an infinite number of computational tasks in one moment.

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Computational Infinity Arising in Non-Convex Optimization Problems

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Keywords. Non-convex optimization; d.c. function; local and global search.

Many optimization problems arising from different application areas turn out to be really non-convex [1], in which, as known, most of local solutions are different from a global one even with respect to the value of the objective function. Moreover, often the number of local solutions increases exponentially w.r.t. the dimension of the space, where the problem is stated.

Nowadays the situation in non-convex optimization may be viewed, as dominated by B&B and its ideological satellites approach.

On the other hand, applying B&B approach often we fall into so-called computational infinity, when the procedure is finite, even we are able to prove the finite convergence of the method, but it is impossible to compute a global solution in a rather reasonable time.

Taking into account the situation we proposed another way for solving d.c. optimization problems the principal step of which will be explained on one of d.c. problems.

The solution methods for d.c. problems are based on three principles [1]–[4].

I. Linearization w.r.t. basic nonconvexities of the problems.

II. Application of most advanced convex optimization methods for solving the linearized problems.

III. Using new mathematical (optimization) tools, as Global Optimality Conditions (GOC) and Global Search Strategy, based on GOC.

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Infinity as Non-Totality

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Keywords. Potential infinity; infinity; transfinity.

The central argument of this article is that infinity as a non-totality finds application, but the notion of transfinity—a completed totality that consists of infinitely many elements—is not required in natural sciences, save the hypothetical case where infinitesimals are required.

Infinity is required in modeling time backwards, where 1) the past is assumed to be never ending, 2) none of the periods of time have existed simultaneously, and thus 3) the periods of time do not form a completed totality. The possibility of a spatially infinite Universe requires that 1') there exists a never ending series of physical objects, 2') all the objects do exists simultaneously, but 3') these objects do not form anything that can be called a physical totality.

I maintain that Aristotle's potential infinity is not adequate for modeling the infinite past nor a spatially infinite Universe: infinity as a non-totality is required in both of these cases, but transfinity is not required. Transfinity does not even fit for modeling the infinite past nor an infinite spatial Universe, because these are not totalities, and a transfinite object is especially a totality. I argue that it is in principle wrong to model a non-totality with something that is a totality. I argue that when infinity is realized in nature, the result cannot be a transfinity realized in nature. This is the view of many physicists who do not consider Relativity as fundamental, such as Neil Turok, Paul Steinhardt and Brian Greene.

I also argue that whenever complete induction is applied in the way that produces a transfinite object, the existence of the resulting object either a) clearly entails the existence of a paradox, or b) it depends on an opinion that does it entail a paradox or not. I argue that the definition of the set theoretic ω fights against The Law of Contradiction, by presenting arguments such as the following: the series of the natural numbers cannot be infinite if each natural number is finite, and grows in each step. Wittgenstein's and Aristotle's views support this idea. I define another sort of a more primitive transfinite object k, that is not troubled with some of the paradoxes of ω. I argue that it depends on the opinion that is k paradoxical or not, and therefore even the genuine existence of k is dubious.

I undermine the above views by the argument that infinitesimals are required under the premisses that 1") the Universe is spatially infinite, and 2") there exists genuine implicate/quantum relations and/or spatiotemporal relations that are supposed to be so fundamental that they affect infinitesimally over an infinite distance. In this special case, even a spatially infinite Universe can be considered as a totality, although in a very narrow sense.

Mathematical Modeling of the Water Retention Curves: The Role of the Menger Sponge

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Keywords. Menger sponge; infinite sets; water retention curve.

Soils as porous media and other compound porous structures are made of a high number of differently sized irregular pores which cover many length scale orders and whose voids play a fundamental role in water flow which can accordingly be considered as dependent on porous media geometry. In the last twenty years the characterization of this geometry has been performed by means of fractal geometry, which allows to relate groundwater properties to soil structural ones. *Sierpinski Carpet* and *Menger Sponge* (MS) geometric structures just fit, respectively in 2- and 3-dimensional space, to describe soil self-similarity fractal behaviour, both for 'solid' and 'pore' phase, and to correlate porosity to water flow. The structure geometry of the micro-porosity can be well defined both by means of percolation theory, which determines the distributions of the single grain size and aggregates (*clusters*) forming the soil and macro-pores and micro-pores tortuosity, and through the MS generalized modeling (*Pore Solid Fractal*, PSF [1]). In any case fractal scaling shows heterogeneity influence on the parameters values. This fact draws attention to the fundamental role played by the *Representative Elementary Volume* (REV) and the best size it should assume at different scales. Even the definition of this problem can require the use of both deterministic and statistical methods, as REV size is strictly connected to grains, aggregates and pores ones, so that in this case both a PSF model based approach, extended to whatever medium with voids and solid components, at different even infinitesimal scales [2], and an approach based on percolation theory and its main principles can be used.

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On Applicability of P-Algorithm for Optimization of Functions Including Infinities and Infinitesimals

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Keywords. Global optimization; arithmetic of infinity.

Some approaches in global optimization theory use models of objective functions to justify rational search strategies for global minimum. Statistical models are used, e. g. in [2, 3]. The P-algorithm axiomatically justified in [4] performs a current observation of the objective function value at the point where the probability of the improvement is maximal; the improvement means observation of a value better than the best observed at previous steps. Recently it has been prove the similarity of the P-algorithm with the radial basis functions algorithm; the latter is based not on a statistical model but on the ideas of interpolation theory. There exist similarities also between the structure of some other algorithms of global optimization. An important class constitute homogenous algorithms. It can be proven that algorithms of this class are applicable to minimize objective functions including infinities and infinitesimals using arithmetic of infinity [1]. In the present paper we prove that P-algorithm is homogeneous.

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For Remarks

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